$\gamma_k$ is the acute angle between the tangent plane at $P_k$ and the $xy$-plane. Then

$$A = \lim_{\|\Delta\|\to 0} \sum_{k=1}^{n} \Delta S_k \sec \gamma_k = \int\int_R \sec \gamma \, dS,$$

provided $\sec \gamma$ is a continuous function of $x$, $y$ in $R$. The direction components of the normal to the surface at a point $(x, y, f(x, y))$ are $f_1(x, y)$, $f_2(x, y)$, $-1$, so that for the acute angle $\gamma$ we have

$$\cos \gamma = \frac{1}{\sqrt{f_1^2 + f_2^2 + 1}},$$

and formula (1) is proved.

Example A. Find the area of a sphere of radius $a$. Take the equation of the hemisphere as

$$z = f(x, y) = \sqrt{a^2 - x^2 - y^2}.$$

Note that $f(x, y)$ does not belong to $C^1$ in the circle $x^2 + y^2 \leqq a^2$. Let us find the area of the surface above the circle $x^2 + y^2 = b^2$, $b < a$, and then let $b \to a$. With obvious notations,

$$\begin{aligned} A_b &= \int\int_{R_b} \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}\, dS \\ &= a \int_0^{2\pi} d\theta \int_0^b \frac{r}{\sqrt{a^2 - r^2}}\, dr \\ &= 2\pi a[a - \sqrt{a^2 - b^2}] \end{aligned}$$

$$\lim_{b\to a-} A_b = 2\pi a^2.$$

The area of the whole sphere is $4\pi a^2$.

## 7.4 Critique of the definition

In view of the student's experience with the definition of arc length of a curve as the limit of the lengths of inscribed polygons, the definition of area in §7.1 may be unexpected. It might seem more natural to consider the area as a limit of areas of inscribed polyhedra. But the latter limit need not exist, even for very simple surfaces. Let us illustrate by a right cylinder of altitude $a$ erected on the circle $x^2 + y^2 = 1$. Its curved surface has area $2\pi a$. Let us inscribe a polyhedron whose faces consist of isosceles triangles as follows. Divide the circumference of each base into $n$ equal arcs subtending angles $\Delta\theta = 2\pi/n$ at the centers, but let the points of subdivision of the top circumference lie midway between those of the bottom circumference. Draw a straight line from each point to its two neighbors on the same circle and to the two nearest points of subdivision on the other circle. The inscribed polyhedron thus formed has $2n$ isosceles triangles for faces. The base of each triangle is $2 \sin (\Delta\theta/2)$ and its altitude, computed by the pythagorean theorem, is

$$c = \sqrt{a^2 + \left(1 - \cos \frac{\Delta\theta}{2}\right)^2}.$$

Hence, the area of the inscribed polyhedron is $2nc \sin (\Delta\theta/2)$.

Next suppose that the number of sides of the polyhedron is increased by first dividing the cylinder into $m$ equal cylinders by planes parallel to the base and then proceeding with each as above. In $c$ we must replace $a$ by $a/m$, and we must note that the total number of faces is now $2mn$. The total area of the inscribed polyhedron is

$$A(m, n) = 2n \sin (\pi/n) \sqrt{a^2 + m^2[1 - \cos (\pi/n)]^2}.$$

Note that

$$\lim_{m\to\infty} A(m, n) = \infty, \qquad \lim_{n\to\infty} A(n, n) = 2\pi a$$

$$\lim_{n\to\infty} A(n^2, n) = 2\pi \sqrt{a^2 + (\pi^4/4)}.$$

Hence, $A(m, n)$ approaches no limit as the number of faces becomes infinite.*

## 7.5 Attraction

Two particles of masses $m_1$ and $m_2$ a distance $r$ apart attract each other, according to the Newtonian law, with a force equal to

$$F = K \frac{m_1 m_2}{r^2},$$

where $K$ is a constant depending upon the units employed. From this law we could set up postulates like those of §6.2 which would continue to have meaning for a continuous distribution of mass. Without giving details, let us find the attraction of a lamina $R$ on a unit particle in that plane but outside $R$. Let $(r, \theta)$ be the polar coordinates of a point of $R$ and let the density of the lamina at that point be $f(r, \theta)$. Assume the particle to be at the pole and compute the component of the attraction $F_x$ in the prime direction. Then with the usual notations we have

$$F_x = \lim_{\|\Delta\|\to 0} \sum_{k=1}^{n} K \frac{f(r_k, \theta_k)}{r_k^2} \cos \theta_k \Delta S_k$$

(2) $$= K \int\int_R \frac{f(r, \theta) \cos \theta}{r^2} \, dS.$$

* This example is due to H. A. Schwarz, *Gesammelte Mathematische Abhandlungen*, Vol. 2, p. 309. Berlin: Julius Springer, 1890.