Math 138A
Winter 2012
R. Schultz

## TAKE HOME ASSIGNMENT

This will be due **Wednesday, February 29, 2012,** at the beginning of the class (1:10 P.M.) unless other arrangements are made in advance. Papers may be left at the Mathematics Department's front desk (Surge 202) by 11:30 that day; if you do so, please let the staff known that the assignment should be placed in my departmental mailbox.

The comments on the final page of this document may helpful in connection with the second and third problems.

**1.** $(i)$ Compute the torsion of the curve $\gamma(t) = (t, t^2, t^4)$; there is a formula for expressing the torsion in terms of an arbitrary parametrization in one of the exercises.

$(ii)$ Let $F$ be a real valued function of two variables defined on an open region $U$ of the coordinate plane such that the gradient $\nabla F$ is never $\mathbf{0}$ on $U$, and let $\gamma(s)$ be a curve with an arc length like parametrization (tangent vector always has length 1) whose image lies in the set of points of $U$ such that $F(x, y) = 0$ and whose curvature is nonzero. Prove that for all values of $s$ the acceleration $\gamma''(s)$ is a scalar multiple of the gradient of $F$ at $\gamma(s)$. [*Hint:* Prove first that $\gamma'(s)$ is perpendicular to the gradient at $\gamma(s)$. What can we say about two nonzero vectors in the plane which are perpendicular to a given nonzero vector, and why is this true?]

**2.** $(i)$ Let $U$ be the set of all points in the coordinate plane except $(0, 0)$, and let $T(u, v)$ be the transformation from $U$ to itself given by

$$(x, y) \quad = \quad T(u, v) \quad = \quad \left( \frac{u}{u^2 + v^2}, \frac{-v}{u^2 + v^2} \right) \ .$$

Compute the Jacobian of $T$ and solve for $u$ and $v$ as functions of $x$ and $y$. [*Hint:* Show that $u^2 + v^2$ can be expressed very simply in terms of $x^2 + y^2$.]

$(ii)$ If $L$ is the line defined by the equation $u + v = 1$, then the image of $L$ under $T$ is contained in a circle. Find an equation in $x$ and $y$ which defines that circle.

**3.** Let $T(u, v)$ be the transformation of the coordinate plane given by

$$(x, y) \quad = \quad T(u, v) \quad = \quad \left( u^2 - v^2, 2uv \right) \ .$$

$(i)$ If $L$ is the horizontal line defined by the equation $v = C$ for some constant $C$, then the image of $L$ under $T$ is a parabola. Find an equation in $x$ and $y$ which defines this parabola.

$(ii)$ Suppose now that $L$ is the line defined by the equation $u + v = 1$. Find a nontrivial equation in $x$ and $y$ which is satisfied by the image of $L$ under $T$. [*Note:* A nontrivial equation is one that is not satisfied by every point in the coordinate plane — for example, $0x + 0y = 0$.]

$(iii)$ Finally, suppose that $L$ is the line defined by the equation $v = u/\sqrt{3}$. Then the image of $L$ under $T$ is contained in some line. Find an equation in $x$ and $y$ which defines this line. Are all points on the line describable as images of points in $L$? Give reasons for your answer.

## Suggestions for working these problems

If we are given a transformation $T$ as in these problems and a curve $\Gamma$ defined by an equation of the form $F(u,v) = 0$, then we can find an equation in $x$ and $y$ defining the image of $\Gamma$ by solving for $u$ and $v$ in terms of $x$ and $y$ and substituting these expressions in $x$ and $y$ into the equation $F(u,v) = 0$.

However, in some cases the best approach is to view the data as a system of three equations in four unknowns

$$x = g(u,v) \;, \quad y = h(u,v), \quad F(u,v) = 0$$

in which we eliminate $u$ and $v$ to get a single equation involving $x$ and $y$. Here is a **typical example:**

*Let $T(u,v)$ be the transformation of the coordinate plane given by*

$$(x,y) = T(u,v) = \left(u^2 + 2v^2, uv\right) \;.$$

*If $L$ is the line $u = v$, describe the image of $L$ under $T$.*

The first step in this sort of problem is to write $u$ in terms of $v$ or vice versa. We can then substitute this into the formulas for $x$ and $y$ to express them entirely in terms of $v$:

$$u = v \implies x = 3v^2 \;, \quad y = v^2$$

Next, we eliminate $v$ from this system of two equations in $x, y, v$ in order to get a single equation involving $x$ and $y$; namely, $x = 3y$. So the image of $L$ is contained in the line with defining equation $x = 3y$. Is the image of $L$ equal to the entire line? **NO,** because $x = v^2$ implies that $x \geq 0$. On the other hand, since every nonnegative real number is a square of a (usually different) real number, it follows that every point on this line satisfying $x \geq 0$ will lie in the image of $L$ under $T$.