SOLUTIONS FOR THE TAKE HOME ASSIGNMENT

1. (i) Compute the torsion of the curve $\gamma(t) = (t, t^2, t^4)$; there is a formula for expressing the torsion in terms of an arbitrary parametrization in one of the exercises.

SOLUTION.

Since the formula in the exercises is off by a sign, answers that are correct except for the sign will receive full credit.

The torsion depends on γ', γ'' and γ''', so the first step is to compute these:

$\gamma'(t) = (1, 2t, 4t^3)$

$\gamma''(t) = (0, 2, 12t^2)$

$\gamma'''(t) = (0, 0, 24t)$

The torsion is given by

$$\tau(t) = \frac{\gamma' \times \gamma'' \cdot \gamma'''}{|\gamma' \times \gamma''|^2}$$

and the numerator is the determinant of the matrix whose rows are γ', γ'' and γ'''. The usual determinant formula shows that the value for the numerator is $48t$.

Computation of the cross product by the standard rule implies that $\gamma' \times \gamma''$ is equal to $(16t^3, -12t^2, 2)$, so that

$$|\gamma' \times \gamma''|^2 = 256t^6 + 144t^4 + 4$$

and if we substitute this and the previously derived formula for $[\gamma', \gamma'', \gamma''']$ we obtain the torsion as a function of t:

$$\tau(t) = \frac{48t}{256t^6 + 144t^4 + 4} = \frac{12t}{64t^6 + 36t^4 + 1}$$

(ii) Let F be a real valued function of two variables defined on an open region U of the coordinate plane such that the gradient ∇F is never $\mathbf{0}$ on U, and let $\gamma(s)$ be a curve with an arc length like parametrization (tangent vector always has length 1) whose image lies in the set of points of U such that $F(x, y) = 0$ and whose curvature is nonzero. Prove that for all values of s the acceleration $\gamma''(s)$ is a scalar multiple of the gradient of F at $\gamma(s)$. [*Hint:* Prove first that $\gamma'(s)$ is perpendicular to the gradient at $\gamma(s)$. What can we say about two nonzero vectors in the plane which are perpendicular to a given nonzero vector, and why is this true?]

SOLUTION.

Following the hint, start by showing $\gamma'(s)$ is perpendicular to the gradient at $\gamma(s)$. By the hypotheses we know that $F(\gamma(s)) = 0$, and if we differentiate this with respect to s and apply the chain rule we obtain the equation

$$0 = \frac{d}{ds} F(\gamma(s)) = \nabla F(\gamma(s)) \cdot \gamma'(s)$$

which means that the tangent vector is perpendicular to the gradient. On the other hand, the assumption of an arc length like parametrization means that $|\gamma'(s)|^2 = 1$, and as usual if we differentiate this with respect to s we obtain the equation $2\,\gamma'(s)\cdot\gamma''(s) = 0$ which is equivalent to $\gamma'(s)\cdot\gamma''(s) = 0$. Hence both $\gamma''(s)$ and $\nabla F(\gamma(s))$ are perpendicular to the unit vector $\gamma'(s)$. Since all these vectors lie in a 2-dimensional vector space, the set of all vectors perpendicular to $\gamma'(s)$ is a 1-dimensional subspace. The condition $\kappa(s) \neq 0$ implies that $\gamma''(s)$ is nonzero, and since they lie in a 1-dimensional subspace we know that each is a nonzero multiple of the other.

2. (*i*) Let U be the set of all points in the coordinate plane except $(0,0)$, and let $T(u,v)$ be the transformation from U to itself given by

$$(x,y) \quad = \quad T(u,v) \quad = \quad \left(\frac{u}{u^2+v^2}, \frac{-v}{u^2+v^2}\right) \ .$$

Compute the Jacobian of T and solve for u and v as functions of x and y. [*Hint:* Show that u^2+v^2 can be expressed very simply in terms of x^2+y^2.]

SOLUTION.

First compute the partial derivatives of the coordinate functions.

$$\frac{\partial x}{\partial u} \quad = \quad \frac{v^2-u^2}{(u^2+v^2)^2}\ , \qquad \frac{\partial y}{\partial u} \quad = \quad \frac{-2uv}{(u^2+v^2)^2}\ , \qquad \frac{\partial x}{\partial v} \quad = \quad \frac{2uv}{(u^2+v^2)^2}\ , \qquad \frac{\partial y}{\partial v} \quad = \quad \frac{v^2-u^2}{(u^2+v^2)^2}$$

This means that the Jacobian is equal to

$$\frac{1}{(u^2+v^2)^4}\cdot\begin{vmatrix} v^2-u^2 & -2uv \\ 2uv & v^2-u^2 \end{vmatrix} \quad = \quad \frac{(v^2-u^2)^2+4uv}{(u^2+v^2)^4} \quad = \quad \frac{(u^2+v^2)^2}{(u^2+v^2)^4} \quad = \quad \frac{1}{(u^2+v^2)^2}\ .$$

Next, solve for u and v as functions of x and y using the hint. One way to start is to express x^2+y^2 in terms of u and v.

$$x^2 \ + \ y^2 \quad = \quad \frac{u^2}{(u^2+v^2)^2} \ + \ \frac{v^2}{(u^2+v^2)^2} \quad = \quad \frac{1}{u^2+v^2}$$

Therefore we also have

$$u^2 \ + \ v^2 \quad = \quad \frac{1}{x^2+y^2}$$

which in turn implies that $x = u(x^2+y^2)$ and $y = -v(x^2+y^2)$, and if we solve for u and v we obtain the equations

$$u \quad = \quad \frac{x}{x^2+y^2}\ , \qquad v \quad = \quad \frac{-y}{x^2+y^2}$$

which means that T is equal to its own inverse transformation!

Note. One can also approach this using simple facts about complex numbers. From this perspective the transformation has the form $T(z) = 1/z$, and $T = T^{-1}$ corresponds to the identity

$$z \quad = \quad \frac{1}{1/z}\ .$$

(*ii*) If L is the line defined by the equation $u+v=1$, then the image of L under T is contained in a circle. Find an equation in x and y which defines that circle.

SOLUTION.

Substitute the expressions for $u(x,y)$ and $v(x,y)$ into the equation $u+v=1$; the resulting equation is

$$1 \quad = \quad u \; + \; v \quad = \quad \frac{x}{x^2+y^2} \; + \; \frac{-y}{x^2+y^2} \; .$$

If we clear this of fractions and ignore the middle equation we obtain the equation $x^2+y^2 \; = \; x-y$, which is equivalent to $x^2-x+y^2+y=0$. If we complete the squares of the quadratic expressions in x and y this equation becomes

$$x^2 - x + \tfrac{1}{4} + y^2 + y + \tfrac{1}{4} \quad = \quad \tfrac{1}{2}$$

which can be rewritten in the form

$$\left(x - \frac{1}{2}\right)^2 \; + \; \left(y + \frac{1}{2}\right)^2 \quad = \quad \frac{1}{2} \; .$$

3. Let $T(u,v)$ be the transformation of the coordinate plane given by

$$(x,y) \quad = \quad T(u,v) \quad = \quad \left(u^2 - v^2, 2uv\right) \; .$$

(i) If L is the horizontal line defined by the equation $v=C$ for some constant C, then the image of L under T is a parabola. Find an equation in x and y which defines this parabola.

SOLUTION.

We have the following system of three equations in x,y,u,v:

$$x \quad = \quad u^2 - v^2 \quad , \qquad y \quad = \quad 2uv \quad , \qquad v \quad = \quad C$$

We need to reduce this to a single equation in x and y by eliminating u and v. The third equation quickly eliminates v, yielding the following system of two equations in x,y,u:

$$x \quad = \quad u^2 - C^2 \quad , \qquad y \quad = \quad 2uC$$

Now solve the second equation for u in terms of y and substitute this expression into the first equation:

$$u \;\; \frac{y}{2C} \quad \Longrightarrow \quad x \quad = \quad \frac{y^2}{4C^2} \; - \; C^2$$

This is the equation for the parabola which is the image of the horizontal line $v=C$.

(ii) Suppose now that L is the line defined by the equation $u+v=1$. Find a nontrivial equation in x and y which is satisfied by the image of L under T. [*Note:* A nontrivial equation is one that is not satisfied by every point in the coordinate plane — for example, $0x+0y=0$.]

SOLUTION.

In this case we obtain the following system:

$$x \quad = \quad u^2 - v^2 \quad , \qquad y \quad = \quad 2uv \quad , \qquad u+v \quad = \quad 1$$

The first step is to solve the third equation for v in terms of u and to substitute this into the first and second equations.

$$v \;=\; 1-u \;\Longrightarrow\; x \;=\; u^2-(1-u)^2 \;=\; 2u-1, \qquad y \;=\; 2u(1-u) \;=\; 2u-2u^2$$

Now eliminate u by solving for u in the first equation and substituting the result into the second:

$$x \;=\; 2u-1 \;\Longrightarrow\; u \;=\; \frac{x-1}{2} \;\Longrightarrow$$

$$y \;=\; (x-1) \;-\; \frac{(x-1)^2}{2} \;=\; -\tfrac{1}{2}x^2+2x-\tfrac{3}{2}$$

(iii) Finally, suppose that L is the line defined by the equation $v = u/\sqrt{3}$. Then the image of L under T is contained in some line. Find an equation in x and y which defines this line. Are all points on the line describable as images of points in L? Give reasons for your answer.

SOLUTION.

In this case we obtain the following system:

$$x \;=\; u^2-v^2 \;, \qquad y \;=\; 2uv \;, \qquad v \;=\; u/\sqrt{3}$$

Eliminate v from the first two equations by substituting the expression for v in the third one:

$$x \;=\; u^2-\frac{u^2}{3}\;\frac{2u^2}{3}\;, \qquad y \;=\; \frac{2u^2}{\sqrt{3}}$$

We can eliminate u^2 from these equations, obtaining the equation $y = x\sqrt{3}$, which is the equation of a line.

The image does is not the entire line because $u^2 \geq 0$ implies $x, y \geq 0$.

Note. It is easy to understand this example in terms of polar coordinates, for the transformation sends the point with polar coordinates (r, θ) into the point with polar coordinates $(r^2, 2\theta)$. In polar coordinates the equation of the original line becomes $\theta = \pi/6$, and the equation of its image becomes $\theta = \pi/3$.