BACKGROUND ON BILINEAR FORMS

Let V be a vector space. A *bilinear form* B is a function of two variables which is scalar (real) valued and has the following properties:

$$B(x, y+z) = B(x,y) + B(x,z)$$
$$B(x+y, z) = B(x,z) + B(y,z)$$
$$B(cx, y) = c \cdot B(x,y) = B(x, cy)$$

all $x, y, z \in V$, $c \in \mathbb{R}$.

We do not assume anything about whether or not $B(x,y) = B(y,x)$ at this point. If B satisfies this condition we shall say B is *symmetric*.

Relation to the Second Fundamental Form.

Suppose $S \subseteq \mathbb{R}^3$ is a smooth surface, $p \in S$ and $T_p(S)$ is the 2-dimensional vector space of tangent vectors to S at p. Then the Second Fundamental Form at p is a bilinear form on $T_p(S)$.

PROPOSITION 1 Let V be a 2-dimensional vector space, let $\{u, v\}$ be a basis for V and let B and B' be bilinear forms on V. Then $B = B'$ if and only if the following hold:

$$B(u,u) = B'(u,u)$$
$$B(u,v) = B'(u,v)$$
$$B(v,u) = B'(v,u)$$
$$B(v,v) = B'(v,v).$$

COROLLARY 2 Let B be as above, then B is symmetric if and only if $B(u,v) = B(v,u)$.

<u>Derivation of Prop. 1:</u> If $B = B'$ then the four equations follow immediately. Suppose now we have the four equations. If $x, y \in V$, write $x = au + bv$, $y = cu + dv$. Then

$$B(x,y) = ac\,B(u,u) + ad\,B(u,v) + bc\,B(v,u) + bd\,B(v,v) = (\text{same with } B' \text{ replacing } B) = B'(x,y) \blacksquare$$

<u>Derivation of Cor. 2:</u> Let $B'(x,y) = B(y,x)$. Then B' is bilinear and the proposition implies that $B = B' \iff B(u,v) = B(v,u)$. But $B = B' \iff B$ is symmetric $\blacksquare$

<u>Application to the Second Fundamental Form:</u>

In this case, suppose we have a local parametrization near p. Then $\frac{\partial \vec{X}}{\partial u}$ & $\frac{\partial \vec{X}}{\partial v}$ form a basis for $T_p(S)$, and by our formulas we see that the Second Fundamental Form at p is symmetric.

Now let $\$_p$ denote the shape operator at p, so that the second fundamental form is equal to $\langle \$_p w_1, w_2 \rangle$ where w_1 & w_2 are in $T_p(S)$ and $\langle --, -- \rangle$ is the usual inner product given by viewing $T_p(S)$ as a vector subspace of $\mathbb{R}^3$. The symmetric nature of the second fundamental form implies that

$$\langle \$_p w_1, w_2 \rangle = \langle \$_p w_2, w_1 \rangle = \langle w_1, \$_p w_2 \rangle.$$

One often states this fact in the form, "The shape operator is self-adjoint."

Digression

We shall discuss self adjointness abstractly.

If V is a real vector space, an inner product on V is a symmetric bilinear form with the following positive definiteness property:

$\langle v,v\rangle \geq 0$ with equality $\Longleftrightarrow$ $v=0$.

<u>Examples</u> Obviously, one is the usual inner product on $\mathbb{R}^2$. More general examples are given by $B((x_1,x_2),(y_1,y_2)) =$

$$x_1y_1 + \frac{d}{2}(x_1y_2 + y_1x_2) + y_2x_2$$

where $0 < d < 2$.

It is an exercise in bookkeeping to check that B is symmetric and bilinear, and in fact $B((x_1,x_2),(x_1,x_2))$ equals

$$x_1^2 + d\,x_1x_2 + x_2^2$$

which we may rewrite as

$$\left(x_1 + \frac{d}{2}x_2\right)^2 + \underbrace{\left(1 - \frac{d^2}{4}\right)}_{\text{positive}} x_2^2.$$

This is a sum of squares and hence is positive; it vanishes $\Longleftrightarrow$ $x_1 + \frac{d}{2}x_2 = 0 = x_2$, which is equivalent to $x_1 = x_2 = 0$.

From linear algebra we know that, if V is n-dimensional with inner product $\langle\, ,\, \rangle$ then V has an orthonormal basis $u_1, \ldots, u_n$; it can be constructed from a given basis using the Gram-Schmidt process.

Now suppose $T : V \to V$ is a linear transformation which is self-adjoint with respect to $\langle\, ,\, \rangle$. The coefficients of the matrix A for T with respect to $\{u_1, \ldots, u_n\}$ are defined by the standard rule:

$$T u_j = \sum a_{ij} u_i .$$

It follows that

$$a_{ij} = \langle T u_j , u_i \rangle$$

and hence if T is self adjoint we have

$$a_{ij} = \langle T u_j, u_i \rangle = \langle u_j, T u_i \rangle = \langle T u_i, u_j \rangle = a_{ji}$$

so that <u>A is a symmetric matrix.</u>