Exercises 40–49 concern the implicit function theorems and the inverse function theorem (Theorems 6.5, 6.6, and 6.7).

40. Let S be described by $z^2y^3 + x^2y = 2$.

(a) Use the implicit function theorem to determine near which points S can be described locally as the graph of a C^1 function $z = f(x, y)$.

(b) Near which points can S be described (locally) as the graph of a function $x = g(y, z)$?

(c) Near which points can S be described (locally) as the graph of a function $y = h(x, z)$?

41. Let S be the set of points described by the equation $\sin xy + e^{xz} + x^3y = 1$.

(a) Near which points can we describe S as the graph of a C^1 function $z = f(x, y)$? What is $f(x, y)$ in this case?

(b) Describe the set of "bad" points of S, that is, the points $(x_0, y_0, z_0) \in S$ where we *cannot* describe S as the graph of a function $z = f(x, y)$.

(c) Use a computer to help give a *complete* picture of S.

42. Let $F(x, y) = c$ define a curve C in $\mathbf{R}^2$. Suppose (x_0, y_0) is a point of C such that $\nabla F(x_0, y_0) \neq \mathbf{0}$. Show that the curve can be represented near (x_0, y_0) as either the graph of a function $y = f(x)$ or the graph of a function $x = g(y)$.

43. Let $F(x, y) = x^2 - y^3$, and consider the curve C defined by the equation $F(x, y) = 0$.

(a) Show that $(0, 0)$ lies on C and that $F_y(0, 0) = 0$.

(b) Can we describe C as the graph of a function $y = f(x)$? Graph C.

(c) Comment on the results of parts (a) and (b) in light of the implicit function theorem (Theorem 6.5).

44. (a) Consider the family of level sets of the function $F(x, y) = xy + 1$. Use the implicit function theorem to identify which level sets of this family are actually unions of smooth curves in $\mathbf{R}^2$ (i.e., locally graphs of C^1 functions of a single variable).

(b) Now consider the family of level sets of $F(x, y, z) = xyz + 1$. Which level sets of this family are unions of smooth surfaces in $\mathbf{R}^3$?

45. Suppose that $F(u, v)$ is of class C^1 and is such that $F(-2, 1) = 0$ and $F_u(-2, 1) = 7$, $F_v(-2, 1) = 5$. Let $G(x, y, z) = F(x^3 - 2y^2 + z^5, xy - x^2z + 3)$.

(a) Check that $G(-1, 1, 1) = 0$.

(b) Show that we can solve the equation $G(x, y, z) = 0$ for z in terms of x and y (i.e., as $z = g(x, y)$) for (x, y) near $(-1, 1)$ so that $g(-1, 1) = 1$.

46. Can you solve

$$\begin{cases} x_2y_2 - x_1 \cos y_1 = 5 \\ x_2 \sin y_1 + x_1 y_2 = 2 \end{cases}$$

for y_1, y_2 as functions of x_1, x_2 near the point $(x_1, x_2, y_1, y_2) = (2, 3, \pi, 1)$? What about near the point $(x_1, x_2, y_1, y_2) = (0, 2, \pi/2, 5/2)$?

47. Consider the system

$$\begin{cases} x_1y_2^2 - 2x_2y_3 = 1 \\ x_1y_1^5 + x_2y_2 - 4y_2y_3 = -9 \\ x_2y_1 + 3x_1y_3^2 = 12 \end{cases}.$$

(a) Show that, near the point $(x_1, x_2, y_1, y_2, y_3) = (1, 0, -1, 1, 2)$, it is possible to solve for y_1, y_2, y_3 in terms of x_1, x_2.

(b) From the result of part (a), we may consider y_1, y_2, y_3 to be functions of x_1 and x_2. Use implicit differentiation and the chain rule to evaluate $\frac{\partial y_1}{\partial x_1}(1, 0)$, $\frac{\partial y_2}{\partial x_1}(1, 0)$, and $\frac{\partial y_3}{\partial x_1}(1, 0)$.

48. Consider the equations that relate cylindrical and Cartesian coordinates in $\mathbf{R}^3$:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta. \\ z = z \end{cases}$$

(a) Near which points of $\mathbf{R}^3$ can we solve for r, θ, and z in terms of the Cartesian coordinates?

(b) Explain the geometry behind your answer in part (a).

49. Recall that the equations relating spherical and Cartesian coordinates in $\mathbf{R}^3$ are

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta. \\ z = \rho\cos\varphi \end{cases}$$

(a) Near which points of $\mathbf{R}^3$ can we solve for ρ, φ, and θ in terms of x, y, and z?

(b) Describe the geometry behind your answer in part (a).

2.7 True/False Exercises for Chapter 2

1. The component functions of a vector-valued function are vectors.

2. The domain of $\mathbf{f}(x, y) = \left(x^2 + y^2 + 1, \frac{3}{x + y}, \frac{x}{y}\right)$ is $\{(x, y) \in \mathbf{R}^2 \mid y \neq 0, x \neq y\}$.