Curvature Exercises

Parabola $y = x^2$.

Parametrize by $\vec{x}(t) = (t, t^2) = (t, t^2, 0)$

Find the curvature at $\vec{x}(t)$.
Use the formula $\kappa(t) = \dfrac{|\vec{x}\,'(t) \times \vec{x}\,''(t)|}{|\vec{x}\,'(t)|^3}$

$\vec{x}\,'(t) = (1, 2t, 0) \qquad |\vec{x}\,'(t)| = \sqrt{1+4t^2}$

$\vec{x}\,''(t) = (0, 2, 0)$.

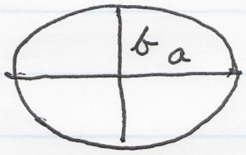
$$\vec{x}\,' \times \vec{x}\,'' = \begin{vmatrix} i & j & k \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = (0, 0, 2).$$

Therefore $\kappa(t) = \dfrac{2}{(1+4t^2)^{3/2}}$

Notice that the maximum curvature occurs when $t = 0$, and as $t \to \infty$ the curvature goes to 0.

Suggestion Analyze the curvature of the hyperbola $y = \frac{1}{x}$, where $x > 0$, similarly.

<u>Ellipse</u> $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad a > b > 0$



Parametrize by
$x = a\cos\theta$
$y = b\sin\theta$

$$\gamma'(\theta) = (-a\sin\theta,\ b\cos\theta)$$
$$\gamma''(\theta) = (-a\cos\theta,\ -b\sin\theta)$$
$$|\gamma'(\theta)| = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta} = \sqrt{(a^2-b^2)\sin^2\theta + b^2}$$

$$\gamma' \times \gamma'' = \begin{vmatrix} i & j & k \\ -a\sin\theta & b\cos\theta & 0 \\ -a\cos\theta & -b\sin\theta & 0 \end{vmatrix} = (0, 0, ab)$$

Hence $\kappa(\theta) = \dfrac{ab}{\sqrt{(a^2-b^2)\sin^2\theta + b^2}^{\,3}}$

At the vertices of the minor axes, where $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ we have $\sin^2\theta = 1$, so the curvature is $\dfrac{ab}{a^3} = \dfrac{b}{a^2}$ (minimum value)

At the vertices of the major axes, where $\theta = 0, \pi$ we have $\sin\theta = 0$, so the curvature is $\frac{ab}{b^3} = \frac{a}{b^2}$ (maximum value).

As one goes around the ellipse, the curvature varies between these values

<u>Twisted cubic</u> $\gamma(t) = (2t, t^2, -\frac{t^3}{3})$

$$\gamma'(t) = (2, 2t, -t^2)$$

$$|\gamma'(t)| = \sqrt{4 + 4t^2 + t^4} = t^2 + 2.$$

$$\gamma''(t) = (0, 2, -2t)$$

$$\gamma' \times \gamma'' = \begin{vmatrix} i & j & k \\ 2 & 2t & -t^2 \\ 0 & 2 & -2t \end{vmatrix} =$$

$(2t^2, 4t, 2)$ – length $= \sqrt{4 + 16t^2 + 4t^4} =$

$2(t^2+2)$. $\kappa(t) = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3} = \frac{2(t^2+2)}{(t^2+2)^3} =$

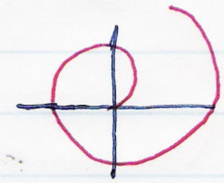
$\frac{2}{(t^2+2)^2}$. MAX $\kappa = 2$ at $t = 0$

AS $t \to \infty$, $\kappa \to 0$.

Here is a more complicated example

Archimedean spiral $\quad r = \theta$ equation

$$\gamma(t) = (t\cos t, t\sin t). \quad t > 0$$



$$\gamma'(t) = (\cos t - t\sin t, \sin t + t\cos t).$$

$$|\gamma'(t)| = \sqrt{1+t^2}$$

How to do this? $\alpha(t) = (\cos t, \sin t)$

$\gamma(t) = t\alpha(t) \quad \alpha'(t) = J\alpha(t) \quad J =$ rotation through $90°$ counterclockwise. Therefore

$$\gamma'(t) = \alpha(t) + t\alpha'(t) = \alpha(t) + tJ\alpha(t)$$

$$\gamma''(t) = 2\alpha'(t) + t\alpha''(t) = 2J\alpha(t) - t\alpha(t)$$

$$[\alpha''(t) = -\alpha(t)]$$

Now $\alpha \times J\alpha = (0,0,1)$ [work it out directly!!]

So $\gamma' \times \gamma'' = (\alpha + tJ\alpha) \times (2J\alpha - t\alpha) =$

$2(\alpha \times J\alpha) - t^2(J\alpha \times \alpha) = (2+t^2)(\alpha \times J\alpha) = (0,0,2+t^2)$

Hence $\kappa(t) = \dfrac{t^2+2}{(1+t^2)^{3/2}}$ $\quad \kappa \to 2$ as $t \to 0$

$\kappa \to 0$ as $t \to \infty$