EXERCISES FOR MATHEMATICS 138A

WINTER 2004

The references denote sections of the text for the course:

M. P. do Carmo, *Differential geometry of Curves and Surfaces*, Prentice-Hall, Saddle River NJ, 1976, ISBN 0–132–12589–7.

I. Classical Differential Geometry of Curves

I.1 : Cross products

(O'Neill, § 2.2)

Additional exercise

1. Verify that the cross product of vectors in $\mathbf{R}^3$ satisfies the *Jacobi identity*:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \ + \ \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) \ + \ \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \ = \ \mathbf{0} \ .$$

I.2 : Parametrized curves

(O'Neill, § 1.4)

O'Neill, pp.21-22: 2, 8

Additional exercises

1. Find a parametrized curve $\alpha(t)$ which traces out the unit circle about the origin in the coordinate plane and has initial point $\alpha(0) = 1$.

2. Let $\alpha(t)$ be a parametrized cure which does not pass through the origin. If $\alpha(t_0)$ is the point in the image that is closest to the origin and $\alpha'(t_0) \neq 0$, show that $\alpha(t_0)$ and $\alpha'(t_0)$ are perpendicular.

3. Two lines are said to be *skew lines* if they are disjoint but not parallel. Prove that the distance between the skew lines $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{u}$ and $\mathbf{y}(t) = \mathbf{y}_0 + t\mathbf{v}$ is given by

$$\rho \ = \ \frac{\mathbf{u} \times \mathbf{v} \cdot \mathbf{r}}{\mathbf{u} \times \mathbf{v}}$$

where $\mathbf{r} = \mathbf{x}_0 - \mathbf{y}_0$. [*Hints:* The shortest distance between the lines is given by a common perpendicular. You may assume the existence of a common perpendicular when working the problem. It might be helpful to let $\mathbf{x}_1$ and $\mathbf{y}_1$ from these lines lie on this common perpendicular.]

4. Prove that a regular smooth curve lies on a straight line if and only if there is a point that lies on all its tangent lines.

I.3 : Arc length and reparametrization

(O'Neill, §§ 1.4, 2.2)

O'Neill, pp. 56–57: 3–5, 10, 11

Additional exercises

1. Prove that a necessary and sufficient condition for the plane $\mathbf{N} \cdot \mathbf{x} = 0$ to be parallel to the line $\mathbf{x} = \mathbf{x}_0 + t \cdot \mathbf{u}$ is for $\mathbf{N}$ and $\mathbf{u}$ to be perpendicular.

2. (a) Given $a > 0$, consider the set of all continuously differentiable real valued functions f on $[0,1]$ such that $f(0) = 0$ and $f(1) = a > 0$. Define $L(f)$ by the formula $L(f) \;=\; \int_0^a |f'(t)|\,dt$. Show that the minimum value of $L(f)$ is a, and if equality holds then f' is everywhere nonnegative. [*Hints:* Since $f' \leq |f'|$ a similar inequality holds for their definite integrals. This inequality of integrals is strict if and only if $f'(t) < |f'(t)|$ for some t, which happens if and only if $f'(t) < 0$ for that choice of t.]

(b) Let ρ, θ and ϕ denote the usual spherical coordinates, and suppose we have a curve on the sphere of radius 1 about the origin with parametric equations of the form

$$\mathbf{x}(t) \quad = \quad (\cos\theta(t)\,\sin\phi(t),\, \sin\theta(t)\,\sin\phi(t),\, \cos\phi(t)\,)$$

for continuously differentiable functions $\theta(t)$ and $\phi(t)$. Prove that the length of this curve is given by the formula

$$\int_a^b \sqrt{(\theta'(t))^2 \;+\; \sin^2\theta(t)\,(\phi\,'(t))^2}\,dt$$

where the curve is defined on $[a,b]$.

(c) Show that among all regular smooth curves $\mathbf{x}$ that are defined on $[0,1]$, have images on the unit sphere, and connect the points $(1,0,0)$ and $(\cos a, \sin a, 0)$ for some $a < \pi$, the curve of shortest length is given by the great circle arc joining the endpoints, and that any other curve with this length is a weak reparametrization of the great circle arc (*i.e.*, if α is the standard great circle arc, then any other curve β must have the form $\beta(t) = \alpha(\,f(t)\,)$, where f is a 1–1 function from $[0,1]$ to $[0,a]$ that is continuously differentiable and satisfies $f' \geq 0$. [*Hints:* Let $\mathbf{y}$ be the curve in the xy-plane obtained from $\mathbf{x}$ by replacing $\phi(t)$ with $\pi/2$; in other words, $\mathbf{y}$ is the perpendicular projection of the original curve onto the xy-plane. Why does the spherical coordinate arc length formula show that the length of $\mathbf{x}$ is greater than or equal to the length of $\mathbf{y}$? And why is there strict inequality if $\phi\,'(t_0)\,\sin\theta(t_0) \neq 0$ somewhere? Why does this mean that the plane curve $(cos\theta(t), \sin\theta(t), 0)$ is a weak reparametrization of $(\cos a\,t, \sin a\,t, 0$? Recall that by continuity the latter implies $\phi\,'(t) \neq 0$ for all t sufficiently close to t_0. What does part (a) imply if ϕ is constant?]

Note. The final part of the problem is a special case of the well known result that the shortest curve on a sphere joining two points is given by the smaller of the arcs on the great circle through the points; in fact, one can use this special case to prove the general statement. [A file containing a detailed proof may be inserted into the course directory eventually.]

I.4 : Curvature and torsion

(O'Neill, § 2.3)

Additional exercises

1. Suppose a curve is given in polar coordinates by $r = r(\theta)$ where $\theta \in [a, b]$.
(i) Show that the arc length is $\int_a^b \sqrt{r^2 + (r')^2}\, d\theta$.
(ii) Show that the curvature is

$$k(\theta) = \frac{2(r')^2 - rr' + r^2}{[r^2 + (r')^2]^{3/2}} \ .$$

2. Let α and β be regular parametrized curves such that β is the arc length reparametrization of α. Let t be the parameter for α and s for β. Prove the following:

(a) $dt/ds = 1/|\alpha'|$, $d^2t/ds^2 = -(\alpha' \cdot \alpha''/|\alpha'|^4$

(b) The curvature is given by

$$k(t) = \frac{\alpha' \times \alpha''}{|\alpha'|^3}$$

(c) The torsion is given by

$$\tau(t) = -\frac{\alpha' \times \alpha'' \cdot \alpha'''}{|\alpha' \times \alpha''|^2}$$

(d) If α has coordinate functions x and y, then the signed curvature of α at t is equal to

$$k(t) = \frac{x'y'' - x''y'}{[(x')^2 + (y')^2]^{3/2}}$$

3. Show that the curvature of a regular parametrized curve α at t_0 is equal to the curvature of the plane curve γ which is the perpendicular projection of α onto the osculating plane of α at t_0.

4. Consider the problem of designing a set of railroad tracks that contains a pair of parallel tracks along with a third going from the first to the second smoothly. Mathematically, the parallel tracks themselves may be viewed as corresponding to the parallel lines $y = 0$ and $y = 1$ in the coordinate plane, and the track going from one to the other may be viewed as a regular smooth curve that is the graph of a twice differentiable function f such that $f(x)$ is zero if $t \leq 0$, $f(x) = 1$ if $t \geq 1$, and on $[0, 1]$ the function f is given by a polynomial $p(x)$. The existence of a second derivative ensures that the slope of the tangent line would be a continuous function of x, and in addition we want to assume that *the curvature is also a continuous function of* x. Find a polynomial $p(x)$ of degree 5 such that all the required conditions are fulfilled. [*Hint:* If we are given a graph curve with parametric equatitons $(t,\ y(t))$, then the curvature at parameter value t is given by the formula

$$k(t) \quad = \quad \frac{|y''|}{\left(1 + (y')^2\right)^{3/2}}$$

and one step in the argument is to use this fact to compute $p''(0)$ and $p''(1)$. In fact, the conditions of the problem uniquely specify the values of p and its first and second derivatives at both 0 and 1. Why does this mean the only values to find are the coefficients of x^3, x^4 and x^5?]

Optional. Graph the function f using calculator or computer graphics.

I.5 : Frenet-Serret Formulas

(O'Neill, §§ 2.3–2.4)

O'Neill, pp. 64–66: 1, 5

Additional exercises

1. Let $\mathbf{x}$ be a regular smooth curve with a continuous third derivative, and let $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ be its Frenet trihedron. Prove that there is a vector $\mathbf{W}$ (the *Darboux vector*) such that $\mathbf{T}' = \mathbf{W} \times \mathbf{T}$, $\mathbf{N}' = \mathbf{W} \times \mathbf{N}$, and $\mathbf{B}' = \mathbf{W} \times \mathbf{B}$. What is the length of $\mathbf{W}$?

2. If $\mathbf{x}$ is defined for $t > 0$ by the formula

$$\mathbf{x}(\mathbf{t}) \quad = \quad \left(t, \ \frac{1+t}{t}, \ \frac{1-t^2}{t} \right)$$

show that $\mathbf{x}$ is planar.

II. Topics from Multivariable Calculus and Geometry

II.1 : Differential forms

(O'Neill, §§ 1.5–1.6)

O'Neill, pp. 25–26: 5, 6 (first part only), 9 (last sentence only)
O'Neill, pp. 31–32: 1, 3–5

Additional exercise

1. Suppose that ω ia a 2-form on $\mathbf{R}^3$ such that $\omega \wedge dx = 0$. Explain why there is a 1-form θ such that $\omega = \theta \wedge dx$.

II.2 : Smooth mappings

(O'Neill, §§ 1.7, 3.2)

Additional exercises

Definition. A subset K of $\mathbf{R}^n$ is said to be *convex* if whenever $\mathbf{x}$ and $\mathbf{y}$ lie in K then the whole line segment defined by the parametrized curve $\mathbf{x} + t\,(\mathbf{y} - \mathbf{x})$ for $t \in [0, 1]$ is contained in K.

1. Prove that an open convex set is a connected domain [*Hint:* Imitate the proof for the set of all point whose distance from some point $\mathbf{p}$ is less than some positive number r.].

2. Show by example that an intersection of two connected domains in $\mathbf{R}^2$ is not necessarily a connected domain. [*Hint:* Let U be the annular region defined by the inequalities $1 < x^2 + y^2 < 9$

and let V be the horizontal strip defined by the inequality $|y| < \frac{1}{2}$. Verify that U is arcwise connected using the polar coordinate mapping, which yields a continuous 1–1 mapping from the convex set $(1,3) \times [0, 2\pi)$ onto U. If $U \cap V$ were connected then by a result in the Appendix to Chapter 5 in do Carmo, it would also be arcwise connected. Suppose now that $\mathbf{x}$ is a curve joining the points (± 2.0). By the Intermediate Value Theorem there must be some parameter value t_0 such that the first coordinate of $\mathbf{x}(t_0)$ is equal to zero. Why does this mean that $\mathbf{x}$ cannot lie entirely inside $U \cap V$?]

3. Given an matrix A with real entries , let $|A|$ denote the Euclidean length given by the square root of the standard sum $\sum_{i,j} |a_{i,j}|^2$. If P and Q are two matrices with real entries such that the product PQ can be defined, prove that $|PQ| \leq |P| \cdot |Q|$.

4. Let U be a convex connected domain in $\mathbf{R}^n$, and let $f : U \to \mathbf{R}^m$ be a smooth $\mathcal{C}^1$ function.

(a) Prove that

$$f(\mathbf{y}) - f(\mathbf{x}) \quad = \quad \int_0^1 \left([Df(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))](\mathbf{y} - \mathbf{x}) \right) dt$$

for all $\mathbf{x}$, $\mathbf{y} \in U$. [*Hint:* Explain why the integrand is the derivative of the function

$$f(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))$$

using the Chain Rule.]

(b) Suppose that the derivative matrix function Df satisfies $|Df| \leq M$ on U. Prove that

$$|f(\mathbf{y}) - f(\mathbf{x})| \quad \leq \quad M \cdot |\mathbf{y} - \mathbf{x}|$$

for all $\mathbf{x}$, $\mathbf{y} \in U$.

Note. An inequality of this sort is called a *Lipschitz condition.*

II.3 : Inverse and Implicit Function Theorems

(O'Neill, § 1.7)

Additional exercises

1. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a $\mathcal{C}^r$ function such that its derivative f' is everywhere positive and the limits of $f(t)$ as $t \to \pm\infty$ are $\pm\infty$ respectively. Prove that f has a $\mathcal{C}^r$ inverse function.

2. Prove that $F(x, y) = (e^x + y,\ x - y)$ defines a 1–1 onto $\mathcal{C}^\infty$ map from $\mathbf{R}^2$ to itself with a $\mathcal{C}^\infty$ inverse.

3. Prove that $F(x, y) = (xe^y + y,\ xe^y - y)$ defines a 1–1 onto $\mathcal{C}^\infty$ map from $\mathbf{R}^2$ to itself with a $\mathcal{C}^\infty$ inverse.

4. (a) Using the change of variables formula, explain briefly why the area of a set in $\mathbf{R}^2$ is the same as the area of its image under a rigid motion of the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where A is a rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(b) More generally, if we are given an arbitrary **affine** transformation as above, where the only condition on A is invertibility, how is the area of a set $\mathcal{F}$ related to the area of its image $T(\mathcal{F})$?

5. A smooth $\mathcal{C}^r$ mapping f from a connected domain $U \subset \mathbf{R}^2$ into $\mathbf{R}^2$ is said to be *regularly conformal* at $\mathbf{p} = (u_0, v_0) \in U$ if the Jacobian of f is positive and for all regular smooth curve pairs $\mathbf{x}$ and $\mathbf{y}$ satisfying $\mathbf{x}(s_0) = \mathbf{y}(s_0) = \mathbf{p}$ the angle between $\mathbf{x}'(s_0)$ and $\mathbf{y}'(s_0)$ is equal to the angle between $[f \circ \mathbf{x}]'(s_0)$ and $[f \circ \mathbf{y}]'(s_0)$.

(a) Prove that the partial derivatives of the coordinate functions satisfy the *Cauchy-Riemann equations*:

$$\frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2}\,, \qquad \frac{\partial f_2}{\partial x_1} = -\frac{\partial f_1}{\partial x_2}$$

[*Hint:* If $A = Df(\mathbf{p})$, one needs to show that $\cos\angle(A\,\mathbf{x},\ A\,\mathbf{y}) = \cos\angle(\mathbf{x},\ \mathbf{y})$ for all nonzero vectore $\mathbf{x}$ and $\mathbf{y}$. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ denote the columns of A, and let J denote counterclockwise rotation through $\pi/2$. Why is $\mathbf{a}_2 = c\,J(\mathbf{a}_1)$ for some constant c, and why does the determinant condition imply c is positive? Explain why $A(\mathbf{e}_1 + \mathbf{e}_2) = \mathbf{a}_1 + \mathbf{a}_2$ must be perpendicular to $A(\mathbf{e}_1 - \mathbf{e}_2) = \mathbf{a}_1 - \mathbf{a}_2$, and use this to conclude that $c = 1$.]

(b) There is a modified version of this relation that holds among the partial derivatives if the Jacobian is **negative**. State it and explain why it is true. [*Hint:* Consider what happens if one composes f with the reflection map $S(x, y) = (x,\ -y)$.]

Note. Functions satisfying the Cauchy-Riemann equations are also known as *complex analytic* functions, and they are the central objects studied in complex variables courses.

II.4 : Congruence of geometric figures

(O'Neill, §§ 3.1, 3.4–3.5)

1. Let F be an isometry of $\mathbf{R}^n$, and let $\mathbf{x}$ and $\mathbf{y}$ be distinct points of $\mathbf{R}^n$ such that $F(\mathbf{x}) = \mathbf{x}$ and $F(\mathbf{y}) = \mathbf{y}$. Suppose that $\mathbf{z}$ is a point on the line joining $\mathbf{x}$ to $\mathbf{y}$ that can be expressed as $\mathbf{z} = t\mathbf{x} + (1-t)\mathbf{y}$ for some scalar t. Prove that $F(\mathbf{z}) = \mathbf{z}$ also holds. [*Hints:* Use the fact that $F(\mathbf{w}) = A(\mathbf{w}) + \mathbf{b}$ for some linear transformation A along with the identity $\mathbf{b} = t\mathbf{b} + (1-t)\mathbf{b}$.]

2. Prove that congruent curves have equal lengths.

III. Surfaces in 3-Dimensional Space

III.1 : Mathematical descriptions of surfaces

(O'Neill, §§ 4.1, 4.8)

O'Neill, pp. 132–133: 1, 4*bc*, 5, 9

Additional exercises

1. Write down equations defining the surfaces given by the following geometric conditions:

(a) The set of points that are equidistant from the point $(0, 0, 4)$ and the xy-plane.

(b) The set of points that are equidistant from the point $(0, 2, 0)$ and the plane defined by the equation $y = -2$.

(c) The set of points that are equidistant from the points $(0, 0, 0)$ and $(1, 0, 0)$.

(d) The set of points for which the sum of the distances to $(\pm 1, 0, 0)$ is equal to 5.

2. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be linearly independent vectors in $\mathbf{R}^3$. Prove that there is a unique sphere containing these three points and $\mathbf{0}$; *i.e.*, show that the system of equations

$$|\mathbf{x} - \mathbf{a}|^2 \quad = \quad |\mathbf{x} - \mathbf{b}|^2 \quad = \quad |\mathbf{x} - \mathbf{c}|^2 \quad = \quad |\mathbf{x}|^2$$

has a unique solution $\mathbf{x}$.

3. Find the inverse map to the stereograpic projection onto $\mathbf{R}^2$ described in Example 5.2 of O'Neill, and show how to cover the sphere by two parametrized pieces.

III.2 : Parametrizations of surfaces

(O'Neill, § 4.2)

Additional exercises

1. Let $f(x, y, z) = (x + y + z - 1)^2$.

(i) What are the critical points and values?

(ii) For which c is the level set for c a regular surface?

(iii) Same questions for xyz^2.

2. Let Σ be a geometric regular smooth surface, let U be a connected domain in $\mathbf{R}^3$ containing Σ, and let $\mathbf{g} : U \to \mathbf{R}^3$ be a smooth 1–1 onto map such that the Jacobian of $\mathbf{g}$ is nowhere zero (hence it has a global inverse), its image is a connected domain, and more generally the image of any connected subdomain of U is also a connected domain. Prove that $\mathbf{g}(\Sigma)$ is also a geometric regular smooth surface.

III.3 : Tangent planes

(O'Neill, § 4.3)

O'Neill, pp. 150–153: 6*bc*, 10

0. Show that the tangent plane is the same at all points along a ruling of a cylinder.

Definition. A surface S is said to be *globally convex* at a point $\mathbf{p}$ if all points of S lie on one of the half planes determined by this tangent plane at $\mathbf{p}$ (*i.e.*, if the equation of the tangent plane is $\mathbf{a} \cdot \mathbf{x} = b$, then the points of the surface are completely contained in the set determined by the inequality $\mathbf{a} \cdot \mathbf{x} \leq b$ or the reverse inequality $\mathbf{a} \cdot \mathbf{x} \geq b$). A surface is said to be strictly globally convex if in addition for each point $\mathbf{p}$ the intersection of S with the tangent plane consists only of the point $\mathbf{p}$.

The surface S is said to be *locally convex* or strictly locally convex at $\mathbf{p}$ if there is an open disk D containing $\mathbf{p}$ such that $S \cap D$ is globally convex or strictly globally convex.

1. Let $\mathbf{X}$ be a parametrized surface defined on a connected domain U, and let $(a, b) \in U$. Define a level function $L(u, v)$ by $L(u, v) = [\mathbf{X}(u, v),\ \mathbf{X}_u(a, b),\ \mathbf{X}_v(a, b)]$ (the vector triple product).

(a) Explain why the surface is locally convex at $\mathbf{p} = \mathbf{X}(a, b)$ if and only if L has a relative maximum or minimum at (a, b) and why the surface is strictly locally convex there if and only if L has a strict relative maximum or minimum.

(b) Why does the gradient of L vanish at (a, b)?

(c) If $H(a, b)$ is the determinant

$$\begin{vmatrix} [\mathbf{X}_{u,u}(a, b),\ \mathbf{X}_u(a, b),\ \mathbf{X}_v(a, b)] & [\mathbf{X}_{u,v}(a, b),\ \mathbf{X}_u(a, b),\ \mathbf{X}_v(a, b)] \\ [\mathbf{X}_{v,u}(a, b),\ \mathbf{X}_u(a, b),\ \mathbf{X}_v(a, b)] & [\mathbf{X}_{v,v}(a, b),\ \mathbf{X}_u(a, b),\ \mathbf{X}_v(a, b)] \end{vmatrix}$$

explain why a surface is **NOT** locally convex at $\mathbf{p}$ if $H(a, b) < 0$. [*Hint:* Why does L have a saddle point at (a, b)?]

(d) In the notation of the preceding part of the problem, show that the surface is strictly locally convex at $\mathbf{p}$ if $H(a, b) > 0$. [*Hint:* Why does L have a strict local maximum or minimum?]

(e) If $\mathbf{X}$ is a graph parametrization of the form $\mathbf{X}(u, v) = (u, v, f(u, v))$, prove that $H(a, b)$ is a 2×2 determinant of a matrix whose entries are the corresponding second partial derivatives of f at (a, b).

(f) Apply the preceding to show that if $p \geq 2$ then the graph of the function

$$z \;=\; (1 \;-\; |x|^p \;-\; |y|^p)^{1/p}$$

is strictly locally convex at all (x, y) such that $|x|^p + |y|^p < 1$. In particular, the case $p = 2$ merely states that the usual sphere is strictly locally convex at each point (in fact, all these surfaces are *globally* strictly convex, but we shall not attempt to prove this). [*Hint:* If $r > 1$, explain why the derivative of $|x|^r$ is equal to $r\,|x|^{r-1}$. There are three cases, depending upon whether x is positive, negative or zero.]

NOTE. By interchanging the roles of the three coordinates in the preceding result one can in fact show that the sets defined by the equations $|x|^p + |y|^p + |z|^p = 1$ are all regular smooth surfaces and are strictly locally convex at all points.

Further study. Graph the intersection of this surface with the xz-plane for $p = 3$ and 4 using calculator or computer graphics. Try this also for larger values of p and describe the limit of these surfaces as $p \to \infty$.

2. Let S be the cylindrical surface given by the parametric equation(s) $\mathbf{X}(u, v) = (u \cos u,\ u \sin u, v)$ for $u \in (\pi/2,\ 9\pi/2)$ and $v \in (-1, 1)$. This is a cylinder generated by the

Archimedean spiral curve in the plane given in polar coordinates by $r = \theta$. Show that S is locally convex at each point but not globally convex at some point in S (for example, at $(2\pi, 0, 0)$). [*Hints:* Use the results of the preceding exercise to show that the surface is locally convex, and draw a sketch to show that there are points of this curve which lie on both sides of the tangent line to the curve at $(2\pi, 0, 0)$. Can you use this to find two points on the curve which lie on opposite sides of the tangent line?]

NOTE. One can modify the example in this exercise to get a surface that is strictly locally convex but not globally convex at $(2\pi, 0, 0)$) by taking $\sin v$ rather than v to be the third coordinate.

3. For each of the following quadric surfaces, determine the sets of points $\mathbf{p}$ where the surface is locally convex and where it is strictly locally convex.

(a) The hyperboloid of two sheets defined by the equation $z^2 - x^2 - y^2 = 1$, where the two pieces are parametrized by $\mathbf{X}(u, v) = (\sinh v \, \cos u, \sinh v \, \sin u, \, \pm \, \cosh v)$.

(b) The hyperboloid of one sheet defined by the equation $x^2 + y^2 - z^2 = 1$, parametrized by $\mathbf{X}(u, v) = (\cosh v \, \cos u, \cosh v \, \sin u, \, \sinh v)$.

(c) The elliptic paraboloid defined by the equation $z = x^2 + y^2$.

(d) The hyperbolic paraboloid defined by the equation $z = y^2 - x^2$.

4. Determine the tangent planes to the surface $x^2 + y^2 - z^2 = 1$ at all points $(x, y, 0)$ and show they are all parallel to the z-axis.

5. Let f be a smooth function. Show that the tangent planes to the surface $z = x f(y/x)$, where $x \neq 0$, all pass through the origin.

6. Show that if all the normals to a connected surface pass through some point, then the surface is part of a sphere.

7. Show that the tangent planes of the common points for the spheres defined by $|\mathbf{x}|^2 = 1$ and $|\mathbf{x} - \mathbf{a}|^2 = 1$ are perpendicular if and only if $|\mathbf{a}|^2 = 2$. How does this generalize if the radius of one sphere is r and the radius of the other sphere is s?

III.4 : The First Fundamental Form

(O'Neill, § 4.6)

Additional exercises

1. Show that the first fundamental form on the surface of revolution

$$\mathbf{X}(u, v) = (f(u) \cos v, \; f(u) \sin v, \; g(v))$$

is given by $f^2 \, dv \, dv + ((f')^2 + (g')^2) \, dt \, dt$.

2. If the first fundamental form on a paramatrized patch has the form $du \, du + f(u, v) \, dv \, dv$, prove that the v-parameter curves cut off equal segments on all u-parameter curves (the former are the curves where the v coordinate is held constant, and the latter are the curves for which the u coordinate is held constant).

3. Compute the first fundamental forms of the following parametrized surfaces where they are regular.

(*i*) The ellipsoid $(a \sin u \cos v, b \sin u \sin v, c \cos u)$.

(*ii*) The elliptic paraboloid $(au \cos v, bu \sin v, u^2)$.

(*iii*) The hyperbolic paraboloid $(au \cosh v, bu \sinh v, u^2)$.

(*iv*) The two sheeted hyperboloid $(a \sinh u \cos v, b \sinh u \sin v, c \cosh u)$.

4. Show that a surface of revolution can be parametrized so that $E = E(v)$, $F = 0$, $G = 1$.

III.5 : Surface area

(O'Neill, § 6.7)

Additional exercises

1. Find the area of the corkscrew surface with parametrization $\mathbf{X}(r, \theta) = (r\ \cos\theta,\ r\ \sin\theta, \theta)$ for $1 \le r \le 2$ and $0 \le \theta \le 2\pi$.

2. Find the area of the parametrized Möbius strip

$$\mathbf{X}(u, v) \quad = \quad (\cos u,\ \sin u, 0)\ +\ v \cdot \big(\cos u\ \cos(u/2),\ \sin u\ \cos(u/2),\ \sin(u/2)\,\big)$$

where $u \in (0, 2\pi)$ and $v \in (-h, h)$ with $0 < h < \frac{1}{2}$. You may view the area as being given by an integral over $[0, 2\pi] \times [-h, h]$.

III.6 : Curves as surface intersections

(O'Neill, ???)

Additional exercises

1. The twisted cubic with parametric equations (t, t^2, t^3) is the intersection of the cylindrical surfaces defined by the equations $z - x^3 = 0$ and $y - x^2 = 0$. What is the angle between the gradients of these functions at the point (x, x^2, x^3)?

2. Show that the parametrized curve $\mathbf{x}(\theta) = (1 + \cos\theta, \sin\theta, 2\ \sin(\theta/2)\,)$ is regular and lies on the sphere of radius 2 about the origin and the cylinder $(x - 1)^2 + y^2 = 1$. Also show that the normal vectors to the two surfaces are linearly independent at the points of intersection if $y \neq 0$.

3. Let f and g be two functions with continuous derivatives defined on the open unit disk $u^2 + v^2 < 1$, and suppose there is a point (a, b) in this open disk where $f(a, b) = c = g(a, b)$, so that the graphs of the surfaces intersect at (a, b, c). Prove that the intersection is transverse if and only if $\nabla f(a, b) \neq \nabla g(a, b)$.

IV. Oriented Surfaces

IV.1 : Normal directions and Gauss maps

(O'Neill, § 4.7)

Additional exercises

1. What are the images of the Gauss maps for the following surfaces? Take the unit normals defined by positive multiples of the corresponding functions' gradients.

(i) The hyperbolic cylinder defined by the equation $xy = 1$.

(ii) The paraboloid of revolution defined by the equation $z = x^2 + y^2$.

2. Let $(\Sigma, \mathbf{N})$ be an oriented surface in $\mathbf{R}^3$. Prove that if the image of the Gauss map for $(\Sigma, \mathbf{N})$ is all of the unit sphere S^2, then every plane in $\mathbf{R}^3$ is parallel to a tangent plane for Σ at one or more of its points. Is the converse true? Prove it or give a counterexample.

IV.2 : The Second Fundamental Form

(O'Neill, § 5.1)

O'Neill, pp. 200–201: 3*bd*

Additional exercise

1. Suppose that Σ is an oriented surface whose Second Fundamental Form is identically zero. Show that (locally) Σ is contained in some plane.

IV.3 : Quadratic forms and adjoint transformations

(O'Neill, ???)

Additional exercises

1. Let A be a symmetric 2×2 matrix.
(i) Show that A has two positive eigenvalues if and only if $a_{1,1}$ and $\det A$ are both positive.
(ii) Show that A has one positive and one negative eigenvalue if and only if $\det A$ is negative.
(iii) Show that A has one zero eigenvalue and one positive eigenvalue if and only if $\det A = 0$ and the trace of A is positive.
(iv) How do the criteria in (i) and (iii) change if *positive* is replaced by *negative* in the condition on eigenvalues?

2. Let A be a symmetric 3×3 matrix, and let B be the 2×2 matrix obtained by deleting the third row and column of A. As indicated in the notes, it follows that A has an orthonormal basis of eigenvectors. Suppose that all of the eigenvectors are positive.
(i) Explain why the determinant of A is positive.
(ii) Explain why B also has positive eigenvalues and hence a positive determinant. [*Hint:* Look at the quadratic form in two variables defined by the symmetric matrix B. Why is it positive except at $(0,0)$, where the value ix 0? What does this mean for the eigenvalues of B?]

Note. A basic result in linear algebra called the *Principal Minors Criterion* gives a converse to the preceding results; in the 3×3 case, it states that if A is a symmetric matrix such that $\det A > 0$, $\det B > 0$ and $a_{1,1} = b_{1,1} > 0$, then all the eigenvalues for A are positive. A proof of this fact is essentially given in the following online document:

`http://math.ucr.edu/ res/linalgnotes.pdf`

The first step is to prove a version of Rayleigh's Principle for 3×3 matrices: The minimum and maximum values of the quadratic form determined by A for vectors of unit length are given by the maximum and minimum eigenvalues. Thus the eigenvalues of the matrix are all positive if and only if the value of the quadratic form is positive for all nonzero choices of variables; when this happens we say that the symmetric matrix A is *positive definite*. One can then combine this equivalence with the arguments on pages 84 and 89–90 in the displayed reference to obtain the conclusion described above and its generalization to symmetric $n \times n$ matrices for all values of n.

IV.4 : Normal, Gaussian and mean curvature

(O'Neill, §§ 5.2–5.3)

O'Neill, pp. 213–216: 3, 7, 16*ab*, 17

Additional exercises

1. Complete the computations of the Gaussian and mean curvatures for the hyperboloids of one and two sheets, the ellipsoid, the hyperbolic and elliptic paraboloids, and the Möbius strip.

2. (a) Suppose that $\mathbf{p}$ is a point on the (oriented) surface Σ at a maximum distance from the origin. Prove that the Gaussian curvature at $\mathbf{p}$ is positive.

(b) Suppose that $\mathbf{p}$ is a point on Σ such that the function on Σ whose x-coordinate assumes a maximum value. Prove that the Gaussian curvature at $\mathbf{p}$ is nonnegative, and give an example to show that it is not necessarily positive. [*Hint:* If M is the maximum value, then all points of the surface lie on one closed side of the plane $x = M$. Why must this be the tangent plane to the surface at $\mathbf{p}$?]

3. Suppose that $\mathbf{p}$ is a common point on two surfaces Σ_1 and Σ_2 such that the normals of the two surfaces at $\mathbf{p}$ are linearly independent. Let C be the curve through $\mathbf{p}$ given by the intersection of Σ_1 and Σ_2. Prove that the curvature κ at $\mathbf{p}$ for this curve satisfies

$$\kappa^2 \sin^2 \alpha \quad = \quad \kappa_1^2 \; + \; \kappa_2^2 \; - \; 2\,\kappa_1\,\kappa_2 \cos \alpha$$

where κ_1 and κ_2 are the normal curvatures of the surfaces in the direction of C at $\mathbf{p}$ and α is the angle between the normals to the surfaces at $\mathbf{p}$.

4. The *Third Fundamental Form* of an oriented surface is defined by

$$\mathbf{III}(\mathbf{x},\, \mathbf{y}\,) \quad = \quad \langle D\,\mathbf{N}(\mathbf{p})](\mathbf{x}),\, D\,\mathbf{N}(\mathbf{p})](\mathbf{y}),\, \rangle \;.$$

Prove that $\mathbf{III} - 2\,H\,\mathbf{II} + K\,\mathbf{I} = 0$ where H and K are the mean and Gaussian curvatures. [*Hint:* If A is a diagonalizable matrix explain why $A^2 - \mathrm{trace}(A)\,A + (\det A)\,I = 0$ and use the fact that if T is a self adjoint linear transformation then $\langle T(\mathbf{x}),\, T(\mathbf{y})\,\rangle = \langle T^2(\mathbf{x}),\, \mathbf{y}\,\rangle$.]

5. Assume that a surface Σ has the property that the principal curvatures $\kappa_\pm$ satisfy $|\kappa_\pm| \leq 1$. Does it also follow that curvature of a curve on Σ also satisfies $|\kappa| \leq 1$?

6. Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.

7. Show that if the mean curvature H is identically zero on Σ and the latter has no planar points, then the Gauss map from Σ to S^2 has the following property:

$$\langle DN_p(w_1),\, DN_p(w_2)\rangle \quad = \quad -k(p)\langle w_1, w_2\rangle$$

for all tangent vectors $w_i \in T_p(\Sigma)$. Show that the above condition implies that the angle of two intersecting curves on S^2 and the angle of their spherical images are equal up to sign.

8. Consider the following parametrized surface, known as *Enneper's surface*:

$$\mathbf{X}(u,v) \quad = \quad \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 = v^2\right)$$

(a) Show that the coefficients of the First Fundamental Form are $E = G = (1+u^2+v^2)^2$ and $F = 0$.

(b) Show that the coefficients of the Second Fundamental Form are $e = -g = 2$ and $f = 0$.

(c) Show that the principal curvatures are $\pm 2/E = \pm 2/G$.

9. Suppose that Σ is a regular surface in $\mathbf{R}^3$ and $F : \mathbf{R}^3 \to \mathbf{R}^3$ is the similarity map sending each $\mathbf{x} \in \mathbf{R}^3$ to $c\mathbf{x}$ where c is a fixed positive real number. Let $\Sigma' = F(\Sigma)$. How are the mean and Gaussian curvatures of Σ and Σ' related?

IV.5 : Special classes of surfaces

(O'Neill, §§ 5.4–5.5)

O'Neill, pp. 222–227: 7