# EXERCISES FOR MATHEMATICS 138A

## WINTER 2010

The references denote sections of the text for the course:

> B. O'Neill, *Elementary Differential Geometry* (Second Edition). Academic Press, San Diego, CA, 1997. ISBN: 0–125–26745–2.

# I. Classical Differential Geometry of Curves

### I.1 : Cross products

(O'Neill, § 2.2)

*Additional exercises*

**1.** Verify that the cross product of vectors in $\mathbb{R}^3$ satisfies the *Jacobi identity*:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \ + \ \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) \ + \ \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \ = \ \mathbf{0} \ .$$

**2.** Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be orthonormal vectors in $\mathbb{R}^3$ such that $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ (cross product). Compute $\mathbf{v} \times \mathbf{w}$ and $\mathbf{w} \times \mathbf{u}$.

*Note.* The preceding result has the following consequence: *Suppose that $T$ is a linear transformation on $\mathbb{R}^3$ which takes the standard unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ to the orthonormal vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ respectively. Then we have $T(\mathbf{x} \times \mathbf{y}) = T(\mathbf{x}) \times T(\mathbf{y})$ for all vectors $\mathbf{x}$,$\mathbf{y}$ in $\mathbb{R}^3$.* — The basic idea is merely that if a linear transformation preserves cross products on a basis, then by the Distributive Law of Multiplication it must preserve all cross products.

### I.2 : Parametrized curves

(O'Neill, § 1.4)

O'Neill, pp.21–22: 2, 8

*Additional exercises*

**1.** Find a parametrized curve $\alpha(t)$ which traces out the unit circle about the origin in the coordinate plane and has initial point $\alpha(0) = (0, 1)$.

**2.** Let $\alpha(t)$ be a parametrized cure which does not pass through the origin. If $\alpha(t_0)$ is the point in the image that is closest to the origin and $\alpha'(t_0) \neq 0$, show that $\alpha(t_0)$ and $\alpha'(t_0)$ are perpendicular.

**3.** If $\Gamma$ is the figure 8 curve with parametrization $\gamma(t) = (3/cost, 2\sin 2t)$, where $0 \leq t \leq 2\pi$, find a nontrivial polynomial $P(x, y)$ such that the image of $\gamma$ is contained in the set of points where

$P(x, y) = 0$. [*Hint:* Recall that $\sin 2t = 2\sin t\cos t$ and $\sin^2 t + \cos^2 t = 1$; the latter implies that $\cos^2 t = \sin^2 t\cos^2 t + i\cos^4 t$.]

**4.** Two objects are moving in the coordinate plane with parametric equations $\mathbf{x}(t) = (t^2 - 2, \frac{1}{2}t^2 - 1)$ and $\mathbf{y}(t) = (t, 5 - t^2)$. Determine when, where, and the angle at which the objects meet.

**5.** Prove that a regular smooth curve lies on a straight line if and only if there is a point that lies on all its tangent lines.

## I.3 : Arc length and reparametrization

(O'Neill, §§ 1.4, 2.2)

O'Neill, pp. 56–57: 3–5, 10, 11

*Additional exercises*

**1.** Prove that a necessary and sufficient condition for the plane $\mathbf{N} \cdot \mathbf{x} = 0$ to be parallel to the line $\mathbf{x} = \mathbf{x}_0 + t \cdot \mathbf{u}$ is for $\mathbf{N}$ and $\mathbf{u}$ to be perpendicular.

**2.** Suppose that $F(x, y)$ is a function of two variables with continuous partial derivatives such that $F(a, b) = 0$ but $\frac{\partial}{\partial y}F(a, b) \neq 0$, and also suppose that $g(x)$ is a function such that the set $F(a, b) = 0$ has the parametrization $y = g(x)$ over the interval $[a - h, a + h]$. Prove that the length of this curve is given by the integral

$$\int_{a-h}^{a+h} \frac{|\nabla F(x, g(x))|}{|F_2(x, g(x))|}\, dx$$

where $F_2$ denotes the partial derivative with respect to the second variable. [*Hint:* Use the implicit differentiation formula for $g$ in terms of the partial derivatives of $F$.]

**3.** (a) Given $a > 0$, consider the set of all continuously differentiable real valued functions $f$ on $[0, 1]$ such that $f(0) = 0$ and $f(1) = a > 0$. Define $L(f)$ by the formula $L(f) = \int_0^a |f'(t)|\, dt$ . Show that the minimum value of $L(f)$ is $a$, and if equality holds then $f'$ is everywhere nonnegative. [*Hints:* Since $f' \leq |f'|$ a similar inequality holds for their definite integrals. This inequality of integrals is strict if and only if $f'(t) < |f'(t)|$ for some $t$, which happens if and only if $f'(t) < 0$ for that choice of $t$.]

(b) Let $\gamma(t)$ be a regular smooth curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ such that $\gamma(0) = \mathbf{0}$ and $\gamma(1)$ is the first unit vector $\mathbf{e}_1$ with first coordinate equal to 1 and the other coordinate(s) equal to zero. Prove that the length of $\gamma$ is at least 1, and equality holds if and only if $\gamma$ is a reparametrization of the straight line segment joining $\gamma(0)$ to $\gamma(1)$. [*Hint:* Write $\gamma = (x, y, z)$ in coordinates, let $\beta = (x, 0, 0)$ and explain why the length of $\beta$ is less than or equal to the length of $\gamma$, with equality if and only if $y = z = 0$. Apply the first part of the problem to show that $x(t)$ defines a reparametrization of the line segment joining the endpoints.

*Note.* The file `greatcircles.pdf` in the course directory proves the corresponding result for curves of shortest length on the sphere; namely, these shortest curves are given by great circle arcs. As noted in the cited document, the argument uses material from later units in this course, and at several points it is "somewhat advanced." An more elementary proof for the distance minimizing property of great circles can be derived fairly quickly from the first theorem in the online document and the standard formula which states that the length of a minor circular arc is equal to the product of the radius of its circle times the measure of its central angle expressed in radians.

**4.** (*a*) If an object is attached to the edge of a circular wheel and the wheel is rolled along a straight line on a flat surface at a uniform speed, then the curve traced out by the object is a **cycloid** (there is an illustration in the file `cyc-curves.pdf`). If the circle has radius $a > 0$ and its center starts at the point with coordinates $(0, a)$, then the object starts at $(0, 0)$ and its parametric equations are given by the classical formula $\mathbf{x}(t) = a \cdot (t - \sin t, 1 - \cos t)$.

Find the length of the cycloid over the parameter values $0 \leq t \leq 2\pi$.

(*b*) In the classical geocentric theory of planetary motion which appears in the *Almagest* of Claudius Ptolemy (*c.* 85–165), there is an assumption that planets travel in curves given by *epicycles*. The simplest examples of these involve circular motion where the center of the circle is moving in a circular path around a second circle (this is similar to the motion of the moon around the earth, which is given by an ellipse while the earth itself is moving around the sun by a larger ellipse; an illustration appears in `cyc-curves.pdf`; in the full theory one also allowed the second circle to move around a third cycle, and so on). A typical example is given by the following formula, in which the first circle has radius $\frac{1}{4}$, the second one is the unit circle about the origin, and the body rotates four times around the small circle as the large circle makes one revolution around its center:

$$\mathbf{x}(t) \quad = \quad (\cos t, \sin t) \; + \; \tfrac{1}{4}\,(\cos 4t, \sin 4t)$$

Find the length of this curve over the parameter values $0 \leq t \leq 2\pi$.

**Notes.** For both parts of these exercises the standard formulas for $|\sin \frac{1}{2}\,\theta\,|$ and $|\cos \frac{1}{2}\,\theta\,|$ may be useful.

## I.4 : Curvature and torsion

(O'Neill, § 2.3)

*Additional exercises*

**1.** Suppose a curve is given in polar coordinates by $r = r(\theta)$ where $\theta \in [a, b]$.

$(i)$ Show that the arc length is $\int_a^b \sqrt{r^2 + (r')^2}\, d\theta$.

$(ii)$ Show that the curvature is

$$k(\theta) = \frac{2(r')^2 - rr' + r^2}{[r^2 + (r')^2]^{3/2}} \ .$$

**2.** Let $\alpha$ and $\beta$ be regular parametrized curves such that $\beta$ is the arc length reparametrization of $\alpha$. Let $t$ be the parameter for $\alpha$ and $s$ for $\beta$. Prove the following:

$(a)$ $dt/ds = 1/|\alpha'|$, $d^2t/ds^2 = -(\alpha' \cdot \alpha''/|\alpha'|^4$

$(b)$ The curvature is given by

$$k(t) = \frac{\alpha' \times \alpha''}{|\alpha'|^3}$$

$(c)$ The torsion is given by

$$\tau(t) = -\frac{\alpha' \times \alpha'' \cdot \alpha'''}{|\alpha' \times \alpha''|^2}$$

$(d)$ If $\alpha$ has coordinate functions $x$ and $y$, then the signed curvature of $\alpha$ at $t$ is equal to

$$k(t) = \frac{x'y'' - x''y'}{[(x')^2 + (y')^2]^{3/2}}$$

**3.** Show that the curvature of a regular parametrized curve $\alpha$ at $t_0$ is equal to the curvature of the plane curve $\gamma$ which is the perpendicular projection of $\alpha$ onto the osculating plane of $\alpha$ at $t_0$.

**4.** Consider the problem of designing a set of railroad tracks that contains a pair of parallel tracks along with a third going from the first to the second smoothly. Mathematically, the parallel tracks themselves may be viewed as corresponding to the parallel lines $y = 0$ and $y = 1$ in the coordinate plane, and the track going from one to the other may be viewed as a regular smooth curve that is the graph of a twice differentiable function $f$ such that $f(x)$ is zero if $t \leq 0$, $f(x) = 1$ if $t \geq 1$, and on $[0, 1]$ the function $f$ is given by a polynomial $p(x)$. The existence of a second derivative ensures that the slope of the tangent line would be a continuous function of $x$, and in addition we want to assume that *the curvature is also a continuous function of* $x$. Find a polynomial $p(x)$ of degree 5 such that all the required conditions are fulfilled. [*Hint:* If we are given a graph curve with parametric equatitons $(t,\, y(t))$, then the curvature at parameter value $t$ is given by the formula

$$k(t) \quad = \quad \frac{|y''|}{(1 + (y')^2)^{3/2}}$$

and one step in the argument is to use this fact to compute $p''(0)$ and $p''(1)$. In fact, the conditions of the problem uniquely specify the values of $p$ and its first and second derivatives at both 0 and 1. Why does this mean the only values to find are the coefficients of $x^3$, $x^4$ and $x^5$?]

*Optional.* Graph the function $f$ using calculator or computer graphics.

### I.5 : Frenet-Serret Formulas

(O'Neill, §§ 2.3–2.4)

O'Neill, pp. 64–66: 1, 5

*Additional exercises*

**1.** Let $\mathbf{x}$ be a regular smooth curve with a continuous third derivative, and let $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ be its Frenet trihedron. Prove that there is a vector $\mathbf{W}$ (the *Darboux vector*) such that $\mathbf{T}' = \mathbf{W} \times \mathbf{T}$, $\mathbf{N}' = \mathbf{W} \times \mathbf{N}$, and $\mathbf{B}' = \mathbf{W} \times \mathbf{B}$. What is the length of $\mathbf{W}$?

**2.** If $\mathbf{x}$ is defined for $t > 0$ by the formula

$$\mathbf{x}(\mathbf{t}) \quad = \quad \left( t, \, \frac{1+t}{t}, \, \frac{1-t^2}{t} \right)$$

show that $\mathbf{x}$ is planar.

## II. Topics from Multivariable Calculus and Geometry

### II.1 : Differential forms

(O'Neill, §§ 1.5–1.6)

O'Neill, pp. 25–26: 5, 6 (first part only), 9 (last sentence only)
O'Neill, pp. 31–32: 1, 3–5

*Additional exercise*

**1.** Suppose that $\omega$ ia a 2-form on $\mathbb{R}^3$ such that $\omega \wedge dx = 0$. Explain why there is a 1-form $\theta$ such that $\omega = \theta \wedge dx$.

### II.2 : Smooth mappings

(O'Neill, §§ 1.7, 3.2)

*Additional exercises*

**Definition.** A subset $K$ of $\mathbb{R}^n$ is said to be *convex* if whenever $\mathbf{x}$ and $\mathbf{y}$ lie in $K$ then the whole line segment defined by the parametrized curve $\mathbf{x} + t\,(\mathbf{y} - \mathbf{x})$ for $t \in [0, 1]$ is contained in $K$.

**1.** ($a$) Prove that an open convex set is a connected domain [*Hint:* Imitate the proof for the set of all point whose distance from some point $\mathbf{p}$ is less than some positive number $r$.].

($b$) Describe an example of a connected domain in the plane which is not convex (you do not need to prove that the domain satisfies these conditions).

**2.** Show by example that an intersection of two connected domains in $\mathbb{R}^2$ is not necessarily a connected domain. [*Hint:* Let $U$ be the annular region defined by the inequalities $1 < x^2 + y^2 < 9$ and let $V$ be the horizontal strip defined by the inequality $|y| < \frac{1}{2}$. Verify that $U$ is arcwise connected using the polar coordinate mapping, which yields a continuous 1–1 mapping from the convex set $(1, 3) \times [0, 2\pi)$ onto $U$. If $U \cap V$ were connected then by a result in the Appendix to Chapter 5 in do Carmo, it would also be arcwise connected. Suppose now that $\mathbf{x}$ is a curve joining the points $(\pm 2. 0)$. By the Intermediate Value Theorem there must be some parameter value $t_0$ such that the first coordinate of $\mathbf{x}(t_0)$ is equal to zero. Why does this mean that $\mathbf{x}$ cannot lie entirely inside $U \cap V$?]

**3.** Given an matrix $A$ with real entries , let $|A|$ denote the Euclidean length given by the square root of the standard sum $\sum_{i,j} \ |a_{i,j}|^2$. If $P$ and $Q$ are two matrices with real entries such that the product $PQ$ can be defined, prove that $|PQ| \leq |P| \cdot |Q|$.

**4.** Let $U$ be a convex connected domain in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^m$ be a smooth $\mathcal{C}^1$ function.

(a) Prove that

$$f(\mathbf{y}) - f(\mathbf{x}) \quad = \quad \int_0^1 \ \big( \big[ Df\,(\mathbf{x} + t\,(\mathbf{y} - \mathbf{x})) \big] (\mathbf{y} - \mathbf{x}) \, \big) \ dt$$

for all $\mathbf{x}$, $\mathbf{y} \in U$. [*Hint:* Explain why the integrand is the derivative of the function

$$f\,(\mathbf{x} + t\,(\mathbf{y} - \mathbf{x}))$$

using the Chain Rule.]

(b) Suppose that the derivative matrix function $Df$ satisfies $|Df| \leq M$ on $U$. Prove that

$$|f(\mathbf{y}) - f(\mathbf{x})| \quad \leq \quad M \cdot |\mathbf{y} - \mathbf{x}|$$

for all $\mathbf{x}$, $\mathbf{y} \in U$.

**Note.** An inequality of this sort is called a *Lipschitz condition.*

## II.3 : Inverse and Implicit Function Theorems

(O'Neill, § 1.7)

*Additional exercises*

**1.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^r$ function such that its derivative $f'$ is everywhere positive and the limits of $f(t)$ as $t \to \pm\infty$ are $\pm\infty$ respectively. Prove that $f$ has a $\mathcal{C}^r$ inverse function.

**2.** Prove that $F(x, y) = (e^x + y, \ x - y)$ defines a 1–1 onto $\mathcal{C}^\infty$ map from $\mathbb{R}^2$ to itself with a $\mathcal{C}^\infty$ inverse.

**3.** Prove that $F(x,y) = (xe^y + y,\, xe^y - y)$ defines a 1–1 onto $\mathcal{C}^\infty$ map from $\mathbb{R}^2$ to itself with a $\mathcal{C}^\infty$ inverse.

**4.** (*a*) Using the change of variables formula, explain briefly why the area of a set in $\mathbb{R}^2$ is the same as the area of its image under a rigid motion of the form $T(\mathbf{x}) = A\,\mathbf{x} + \mathbf{b}$, where $A$ is a rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(*b*) More generally, if we are given an arbitrary **affine** transformation as above, where the only condition on $A$ is invertibility, how is the area of a set $\mathcal{F}$ related to the area of its image $T(\mathcal{F})$?

**5.** A smooth $\mathcal{C}^r$ mapping $f$ from a connected domain $U \subset \mathbb{R}^2$ into $\mathbb{R}^2$ is said to be *regularly conformal* at $\mathbf{p} = (u_0,\, v_0) \in U$ if the Jacobian of $f$ is positive and for all regular smooth curve pairs $\mathbf{x}$ and $\mathbf{y}$ satisfying $\mathbf{x}(s_0) = \mathbf{y}(s_0) = \mathbf{p}$ the angle between $\mathbf{x}'(s_0)$ and $\mathbf{y}'(s_0)$ is equal to the angle between $[f \circ \mathbf{x}]'(s_0)$ and $[f \circ \mathbf{y}]'(s_0)$.

(*a*) Prove that the partial derivatives of the coordinate functions satisfy the *Cauchy-Riemann equations*:

$$\frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2}\,, \qquad \frac{\partial f_2}{\partial x_1} = -\frac{\partial f_1}{\partial x_2}$$

[*Hint:* If $A = Df(\mathbf{p})$, one needs to show that $\cos \angle(A\,\mathbf{x},\ A\,\mathbf{y}) = \cos \angle(\mathbf{x},\ \mathbf{y})$ for all nonzero vectore $\mathbf{x}$ and $\mathbf{y}$. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ denote the columns of $A$, and let $J$ denote counterclockwise rotation through $\pi/2$. Why is $\mathbf{a}_2 = c\,J(\mathbf{a}_1)$ for some constant $c$, and why does the determinant condition imply $c$ is positive? Explain why $A(\mathbf{e}_1 + \mathbf{e}_2) = \mathbf{a}_1 + \mathbf{a}_2$ must be perpendicular to $A(\mathbf{e}_1 - \mathbf{e}_2) = \mathbf{a}_1 - \mathbf{a}_2$, and use this to conclude that $c = 1$.]

(*b*) There is a modified version of this relation that holds among the partial derivatives if the Jacobian is **negative**. State it and explain why it is true. [*Hint:* Consider what happens if one composes $f$ with the reflection map $S(x,y) = (x,\, -y)$.]

**Note.** Functions satisfying the Cauchy-Riemann equations are also known as *complex analytic* functions, and they are the central objects studied in complex variables courses.

**6.** For what values of $(\rho, \theta, \phi)$ does the spherical coordinate mapping

$$(x,y,z) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$$

satisfy the Jacobian condition in the Inverse Function Theorem? Explain why the complement of this set is a line through the origin.

## II.4 : Congruence of geometric figures

(O'Neill, §§ 3.1, 3.4–3.5)

**1.** Let $F$ be an isometry of $\mathbb{R}^n$, and let $\mathbf{x}$ and $\mathbf{y}$ be distinct points of $\mathbb{R}^n$ such that $F(\mathbf{x}) = \mathbf{x}$ and $F(\mathbf{y}) = \mathbf{y}$. Suppose that $\mathbf{z}$ is a point on the line joining $\mathbf{x}$ to $\mathbf{y}$ that can be expressed as $\mathbf{z} = t\mathbf{x} + (1-t)\mathbf{y}$ for some scalar $t$. Prove that $F(\mathbf{z}) = \mathbf{z}$ also holds. [*Hints:* Use the fact that $F(\mathbf{w}) = A(\mathbf{w}) + \mathbf{b}$ for some linear transformation $A$ along with the identity $\mathbf{b} = t\mathbf{b} + (1-t)\mathbf{b}$.]

**2.** Prove that congruent curves have equal lengths.

**3.** A *similarity transformation* of $\mathbb{R}^n$ is a 1–1 onto mapping of the form $T(\mathbf{x}) = cA\mathbf{x} + \mathbf{b}$, where $c > 0$, $A$ is given by an orthogonal $n \times n$ matrix, and $\mathbf{b}$ is some vector in $\mathbb{R}^n$. If $T$ is a *proper* similarity in the sense that $c \neq 1$ (so that $T$ is not an isometry), then prove that there is a unique vector $\mathbf{v}$ such that $T(\mathbf{v}) = \mathbf{v}$. [*Hint:* This is equivalent to showing that there is a unique solution to the equation $(cA - I)\mathbf{x} = \mathbf{b}$, and a uinque solution of this equation exists if and only if the matrix $cA - I$ is invertible. Why is the latter equivalent to showing that $c^{-1}$ is not an eigenvalue of $A$, and why do the orthogonality condition on $A$ and $c \neq 1$ imply this fact?]

**4.** $(a)$ An invertible $n \times n$ matrix $A$ is said to be *conformal* if it preserves angles; *i.e.*, if $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbf{R}^n$ then

$$\cos \angle(\mathbf{x}, \mathbf{y}) \quad = \quad \cos \angle(A\mathbf{x}, A\mathbf{y})$$

where the cosine may defined by the usual inner product formula.

$(b)$ Suppose that $A = CB$ where $B$ is orthogonal and $c > 0$. Show that $A$ is conformal.

$(c)$ Suppose that $A$ is conformal. Prove that the columns of $A$ are perpendicular. [*Hint:* They define the vectors $A\mathbf{e}_i$ where the $\mathbf{e}_i$ are the standard unit vectors in $\mathbf{R}^n$.]

$(d)$ Suppose that $L_i$ is the (positive length of $A\mathbf{e}_i$. Compute $L_i/L_1$. [*Hint:* look at the angle between $\mathbf{e}_1 + \mathbf{e}_i$ and $\mathbf{e}_1$ and the angle between the images of these vectors under $A$.]

$(e)$ Why do the preceding two parts of the problem imply that if $A$ is conformal then $A = cB$ where $B$ is orthogonal and $c > 0$?

$(f)$ Let $f$ be the map from $\mathbb{R}^2$ to itself defined by $f(u, v) = (u^2 - v^2, 2uv)$. Prove that $Df(u, v)$ is conformal for all $(u, v) \neq (0, 0)$. Do the same for $g(u, v) = (u^3 - 3uv^2, 3u^2v - v^3)$. For both of these exercises, it is helpful to use the Cauchy-Riemann equations from a previous exercise.

$(g)$ Prove that a similarity transformation is conformal.