2×2 DIAGONALIZATION

THEOREMS

We shall need the following results:

THEOREM 1. Let A be a symmetric 2×2 matrix. Then there are orthonormal 2×1 column vectors U and V such that U, V are orthonormal and $AU = c_1 U$, $AV = c_2 V$ for suitable real constants c_1 + c_2.

THEOREM 2. Let V be a 2-dimensional real vector space with inner product $\langle\, , \rangle$, and let $T: V \to V$ be a self-adjoint linear transformation. Then there is an orthonormal basis $\{u, v\}$ of V and constants $c_1, c_2 \in \mathbb{R}$ so that $Tu = c_1 u$ and $Tv = c_2 v$.

Proof of Theorem 1

$\boxed{q = \text{scalar}}$

We begin with some standard observations, each equivalent to the next:

1. There is a nonzero (eigen)vector X such that $AX = qX$. $\boxed{\text{a } 2\times 1 \text{ mtx.}}$
2. There is a nonzero solution to the matrix equation $(A - qI)X = 0$
3. $A - cI$ is not invertible.
4. $\det(A - qI) = 0$
5. q is a root of the polynomial
   $\det(A - tI) = t^2 - (a+d)t + \det A = 0$

   where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

None of these require A to be symmetric.

If we now assume A is symmetric, then $b = c$. If $b = c = 0$ then clearly $\binom{1}{0}$ and $\binom{0}{1}$ form an orthonormal basis of eigenvectors, so let's assume $b = c \neq 0$.

CLAIM Under these conditions, the polynomial $\det(A - tI)$ has two unequal real roots, say r_1 and r_2.

PROOF OF CLAIM We need to show that "$B^2 - 4AC$" is positive in the quadratic polynomial. But this is given by

$$(a+d)^2 - 4(ad - b^2) =$$

$$(a-d)^2 + 4b^2 \geq 4b^2 > 0 \text{ since } b \neq 0.$$

Let X_1 & X_2 be the nonzero vectors associated to r_1 & r_2. First, we claim X_1 & X_2 are perpendicular. To see this, look at

$$r_1(X_1 \cdot X_2) = (AX_1)\cdot X_2 \overset{*}{=} X_1 \cdot AX_2 =$$

(*check this using the fact that A is symmetric)

$$X_1 \cdot r_2 X_2 = r_2(X_1 \cdot X_2).$$

Now $r_1 \neq r_2$, so we get $0 = (r_1 - r_2)(X_1 \cdot X_2)$ which, after division by $r_1 - r_2$, yields $0 = X_1 \cdot X_2$.

To complete the proof, take U and V to be the unit vectors $|X_1|^{-1} \cdot X_1$ and $|X_2|^{-1} \cdot X_2$.

We now proceed to Theorem 2.

Let $U = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $V = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, and

set $w_1 = y_1 u_1 + y_2 u_2$ where $\{u_1, u_2\} =$
$w_2 = z_1 u_1 + z_2 u_2$ $\{u, v\}$ from before.

<u>Bookkeeping</u>: Check that $\{w_1, w_2\}$ is orthonormal using the orthonormality of $\{U, V\}$ and $\{u_1, u_2\}$.

Next, $Tw_1 = y_1 Tu_1 + y_2 Tu_2 =$
$y_1(au_1 + bu_2) + y_2(bu_1 + du_2) =$
$(ay_1 + by_2)u_1 + (by_1 + dy_2)u_2$

But $AU = r_1 U$, so

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} r_1 y_1 \\ r_1 y_2 \end{pmatrix},$$ so the RHS

is just $r y_1 u_1 + r y_2 u_2 =$ ~~rw~~$_1$.
$r(y_1 u_1 + y_2 u_2) = r w_1$.

A similar argument verifies that

$T w_2 = r_2 w_2$.