Comparison of wedge and cross products

The fourth problem in Examination 1 requires a formula relating the wedge product of forms and the cross product of vector fields that was mentioned in the lectures but not written down explicitly. For the sake of completeness, we state the result formally here.

As indicated in Section II.1 of the lecture notes, if $\mathbf{F}$ is a smooth vector field defined on some connected domain in $\mathbf{R}^3$ and $\mathbf{F}$ has coordinate functions given by L, M, and N, then we can define a 1-form $\omega_{\mathbf{F}}$ corresponding to $\mathbf{F}$ by the formula

$$\omega_{\mathbf{F}} \quad = \quad L\,dx \; + \; M\,dy \; + \; N\,dz \; .$$

and if $\mathbf{G}$ is a second vector field with coordinate functions given by P, Q, and R, then the 1-form $\omega_{\mathbf{G}}$ corresponding to $\mathbf{G}$ is given by the formula

$$\omega_{\mathbf{G}} \quad = \quad P\,dx \; + \; Q\,dy \; + \; R\,dz \; .$$

Taking the wedge product of these forms, we obtain the 2-form $\omega_{\mathbf{F}} \wedge \omega_{\mathbf{G}}$.

As noted in the same section of the lecture notes, there is also a different but related 1–1 correspondence between 2-forms and vector fields, in this case sending a vector field $\mathbf{H}$ with coordinate functions A, B, C to the following 2-form:

$$\theta_{\mathbf{H}} \quad = \quad A\,dy \wedge dz \; + \; B\,dz \wedge dx \; + \; C\,dx \wedge dy$$

Combining this with the previous paragraph, we have $\omega_{\mathbf{F}} \wedge \omega_{\mathbf{G}} = \theta_{\mathbf{H}}$ for some vector field $\mathbf{H}$. One would expect that $\mathbf{H}$ can be somehow described in terms of $\mathbf{F}$ and $\mathbf{G}$, and in fact we have the following identity:

COMPATIBILITY RELATIONSHIP. *In the situation above we have* $\omega_{\mathbf{F}} \wedge \omega_{\mathbf{G}} = \theta_{\mathbf{H}}$ *where* $\mathbf{H} = \mathbf{F} \times \mathbf{G}$.

The derivation of this formula is straightforward. One simply expands the left hand side using the definitions of the 1-forms, and then one combines like terms to rewrite this 2-form as $A\,dy \wedge dz + B\,dz \wedge dx + C\,dx \wedge dy$ and notices that A, B and C are merely the coordinates of the cross product $\mathbf{F} \times \mathbf{G}$.■