NAME: ______________________________

Mathematics 138A–001, Winter 2010, Examination 1

Answer Key

1. [*15 points*] Suppose that $\gamma(t) = (t, \frac{2}{3}t^{3/2})$ where $1 \leq t \leq 2$, and let $s(t)$ be the arc length reparametrization. Describe $s(t)$ explicitly as a function of t.

SOLUTION

The general formula is

$$s(t) \quad = \quad \int_1^t \sqrt{x'^2 + y'^2}\ dt$$

and from the definition we know that the expression inside the square root sign is $1 + t$. Therefore we have

$$s(t) \quad = \quad \int_1^t \sqrt{1+u}\, du \quad = \quad \frac{2}{3}\,(1+t)^{3/2}\,\Big|_1^t \quad = \quad \frac{2}{3}\left(\,(1+t)^{3/2} - 2^{3/2}\,\right)\,.$$

2. [*25 points*] Let $\gamma(s)$ be a smooth curve in coordinate 3-space which has a modified arc length parametrization such that $|\gamma'(s)| = 1$ always. Define the curvature of γ at $s = s_0$, and in cases where this curvature is nonzero define the principal normal, binormal and torsion of the curve.

SOLUTION

The curvature is just $|\gamma''(s_0)|$; if this number is nonzero, then $\gamma''(s_0) \neq \mathbf{0}$ and $\mathbf{N}(s_0)$ is the unit vector $|\gamma''(s_0)|^{-1} \cdot \gamma''(s_0)$. The binormal is just $\mathbf{B}(s_0) = \mathbf{T}(s_0) \times \mathbf{N}(s_0)$, where $\mathbf{T} = \gamma'$ in this situation, and the torsion is equal to $-\mathbf{B}'(s_0) \cdot \mathbf{N}(s_0)$.

3. [*10 points*] Suppose that $\alpha(t)$ defines the circle with equation $x^2 + y^2 = 4$ in the plane and $\beta(u)$ defines the circle with equation $(x+1)^2 + y^2 = 9$. Let K_1 and K_2 be the respective curvatures of these circles at the point $(2, 0)$. Evaluate the quotient K_1/K_2. State any results about the curvature of a circle that you use (but a derivation is not necessary).

SOLUTION

The α circle and the β circle have radii equal to 2 and 3 respectively, so the curvatures of these circles are equal to the reciprocal values of $\frac{1}{2}$ $(= K_1)$ and $\frac{1}{3}$ $(= K_2)$. Therefore $K_1/K_2 = 3/2$.

4. [*25 points*] Let $\mathbf{F}(u,v,w) \;=\; \big(\,u(1-v),\, uv(1-v),\, uvw\,\big)$, so that $\mathbf{F}(1,2,3) = (-1,-2,6)$. Prove that for all (x,y,z) sufficiently close to $(-1,-2,6)$ the system of nonlinear equations $(x,y,z) \;=\; \mathbf{F}(u,v,w)$ can be solved for u, v and w.

SOLUTION

By the Inverse Function Theorem, the assertion in the second sentence will be true if we have

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}(1,2,3) \quad \neq \quad 0\ .$$

The general formula for the Jacobian is given by

$$\left|\begin{array}{ccc} 1-v & v(1-v) & vw \\ -u & u-2uv & uw \\ 0 & 0 & uv \end{array}\right|$$

and if we evaluate this at $(u,v,w) = (1,2,3)$ we see that the value of the Jacobian at the given point is equal to

$$\left|\begin{array}{ccc} -1 & -2 & 6 \\ -1 & -3 & 3 \\ 0 & 0 & 2 \end{array}\right| \quad = \quad 2$$

so that the Jacobian at $(1,2,3)$ is nonzero and hence for all (x,y,z) sufficiently close to $(-1,-2,6)$ the system of nonlinear equations $(x,y,z) \;=\; \mathbf{F}(u,v,w)$ can be solved for u, v and w.

5. [*25 points*] Let A denote the 2×2 rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $0 < \theta < 2\pi$, and let $T(\mathbf{x})$ be the isometry $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where $\mathbf{b}$ is some vector in the coordinate plane. Prove that there is a unique vector $\mathbf{z}$ such that $T(\mathbf{z}) = \mathbf{z}$. [*Hint:* What does one know about solutions to the system of linear equations $(A - I)(\mathbf{x}) = -\mathbf{b}$ for the given values of θ?]

SOLUTION

The condition $T(\mathbf{x}) = \mathbf{x}$ translates into the equation $A\mathbf{x} + \mathbf{b} = \mathbf{x}$, which after rearrangement is in turn is equivalent to $(A - I)(\mathbf{x}) = -\mathbf{b}$. Thus we need to show that the latter has a unique solution for the given choice of A.

By linear algebra, a unique solution exists if and only if $A - I$ is invertible, or equivalently when $\det(A - I) \neq 0$. Direct computation shows that $\det A = 2 - 2\cos\theta$. Since $\cos\theta < 1$ if $0 < \theta < 2\pi$, the determinant is positive for all such choices of θ, and therefore in all cases we know that the given vector equation has a unique solution.