NAME: ______________________________

Mathematics 138A–001, Winter 2010, Final Examination

Answer Key

1. [*25 points*] Find the curvature and torsion of the helix curve with parametrization $\gamma(t) = (\cos t, \sin t, t)$. [*Hint:* A modified arc length parametrization is given by $s = \sqrt{2}\,t$ or equivalently $t = s/\sqrt{2}$.]

SOLUTION

Take s as in the hint. Then the unit tangent vector is given by

$$\mathbf{T}(s) \quad = \quad \gamma'(s) \quad = \quad \frac{d}{ds}\big(\cos(s/\sqrt{2}),\, \sin(s/\sqrt{2}),\, (s/\sqrt{2})\big) \quad =$$

$$\big(-(1/\sqrt{2})\sin(s/\sqrt{2}),\, (1/\sqrt{2})\cos(s/\sqrt{2}),\, (1/\sqrt{2})\big)$$

so that

$$\mathbf{T}'(s) \quad = \quad -\frac{1}{2}\cdot\big(\cos(s/\sqrt{2}),\, \sin(s/\sqrt{2}),\, 0\big)\ .$$

The length of this vector is the curvature, and it is equal to $\frac{1}{2}$; the principal normal $\mathbf{N}$ is the unit vector in the same direction and hence is $-(\cos(s/\sqrt{2}),\, \sin(s/\sqrt{2}),\, 0)$. It follows that the binormal $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ must be

$$\big((1/\sqrt{2})\sin(s/\sqrt{2}),\, -(1/\sqrt{2})\cos(s/\sqrt{2}),\, (1/\sqrt{2})\big)$$

which implies $\mathbf{B}'(s) = -\frac{1}{2}\mathbf{N}(s)$. By the definition of torsion, it follows that the torsion must be $\frac{1}{2}$.

2. [*25 points*] Let $\mathbf{F}(u,v) = (u^2 + v^2, v^2 - u^2)$. Compute the derivative matrix $D\mathbf{F}(u,v)$. Find a point (a,b) with integral coordinates for which the derivative matrix $D\mathbf{F}(a,b)$ is zero, and find a point (c,d) with integral coordinates for which the rank of the derivative matrix $D\mathbf{F}(c,d)$ is equal to 1 (in other words, it is not invertible but still nonzero).

SOLUTION

The derivative matrix is given by

$$D\mathbf{F}(u,v) \quad = \quad \begin{pmatrix} 2u & 2u \\ -2v & 2v \end{pmatrix}$$

and this matrix is zero if and only if $(u,v) = (0,0)$. Likewise this matrix has rank 1 if and only if it is nonzero but its Jacobian is nonzero; since the Jacobian is $8uv$, it vanishes if and only if $u = 0$ or $v = 0$. Thus it has rank 1 at every nonzero point of the form $(a,0)$ or $(0,b)$, and it is enough to take a or b equal to some nonzero integer.

3. [*20 points*] Let S be the ruled surface with parametrization

$$\mathbf{X}(u,v) \;=\; (t, e^t, 0) + u(0, \cos t, \sin t)\ .$$

Compute the First Fundamental Form of S with respect to this parametrization.

SOLUTION

The first and second partial derivatives of $\mathbf{X}$ are given by

$$\mathbf{X}_1 \;=\; (1, e^t - u\sin t, u\cos t)\ \ , \quad \mathbf{X}_2 \;=\; (0, \cos t, \sin t)$$

so the coefficients of the First Fundamental Form are equal to $E = \mathbf{X}_1 \cdot \mathbf{X}_1 = 1 + u^2 + e^{2t} - 2e^t u \sin t$, $F = \mathbf{X}_1 \cdot \mathbf{X}_2 = 0$, and $G = \mathbf{X}_2 \cdot \mathbf{X}_2 = 1$. Hence the First Fundamental Form must be

$$(1 + u^2 + e^{2t} - 2e^t u \sin t)\, dt\, dt \;+\; du\, du\ .$$

4. [*25 points*] Let $\gamma(t)$ be a regular smooth curve with parametrization $\big(\,x(t), y(t)\,\big)$, and let S be the cylindrical surface with parametrization $\mathbf{X}(t,u) = \big(\,x(t), y(t), u\,\big)$. Show that the Gauss map is constant on the vertical lines in S of the form $\big(\,x(a), y(a), u\,\big)$, where a is some fixed value of the first variable and u is arbitrary.

Extra credit. [*20 points*] Using this, show that at every point $\big(\,x(a), y(a), b\big)$ on S the Shape Operator for S has determinant equal to zero.

SOLUTION

Let $\Omega(t,u) = \mathbf{X}_1(t,u) \times \mathbf{X}_2(t,u)$, so that the orientation $\mathbf{N}$ is given by $\varepsilon|\Omega|^{-1} \cdot \Omega$, where $\varepsilon = \pm 1$. We need to choose an orientation, and we shall take the orientation with the positive sign (everything works similarly for the other choice).

We have $\mathbf{X}_1(t,u) = \big(\,x'(t), y'(t), 0\,\big)$ and $\mathbf{X}_2(t,u) = (0,0,1)$, and therefore

$$\Omega(t,u) = \mathbf{X}_1(t,u) \times \mathbf{X}_2(t,u) \quad = \quad \big(\,y'(t), -x'(t), 0\,\big)$$

which means that Ω, and hence also $\mathbf{N}$, will only depend upon t. But this means that $\mathbf{N}$ is constant on the vertical lines in S of the form $\big(\,x(a), y(a), u\,\big)$.

Extra credit question. The shape operator $\mathbf{S}$ is given in terms of the parametrization by $-D\mathbf{N}(t,u)$, and the shape operator has a zero determinant at $\mathbf{p} = \mathbf{X}(t_0, u_0)$ if and only if

$$-D\mathbf{N}\big(\,\beta(v_0)\,\big)[\beta'(v_0)] \quad = \quad \frac{d}{dv}\mathbf{N}\big(\,\beta(v_0)\,\big) \quad = \quad \mathbf{0}$$

for some regular smooth curve β in the surface with $\beta(v_0) = (t_0, u_0)$. By the main part of the problem, the vertical line curves satisfy this condition.

5. [*30 points*] Let S be the graph of the function $f(x,y) = x^2+y^2$, take the standard parametrization $\mathbf{X}(u,v) = (u, v, u^2+v^2)$, and choose the orientation $\mathbf{N}$ which is a positive multiple of $(-2u, -2v, 1)$. Find the Second Fundamental Form of S with respect to this parametrization, compute the Gaussian curvature of S as a function of u and v, and explain why the formula shows that the Gaussian curvature of S is always positive. You may use the fact that the First Fundamental Form is equal to

$$(1+4u^2)\, du\, du \ + \ 8uv\, du\, dv \ + \ (1+4v^2)\, dv\, dv \ .$$

SOLUTION

Here is what we are given:

$$\mathbf{N} \ = \ \frac{1}{\sqrt{1+4u^2+4v^2}} \cdot (-2v, -2u, 1)\ , \quad E \ = \ 1+4u^2\ , \quad F \ = \ 4uv\ , \quad G \ = \ 1+4v^2$$

To simplify notation let $\alpha = \sqrt{1+4u^2+4v^2}$ so that $\mathbf{N} = \alpha^{-1}(-2v, -2u, 1)$; note that $\alpha \geq 1$ everywhere.

To compute the coefficients of the Second Fundamental Form, we need the second partial derivatives of $\mathbf{X}$, which are $\mathbf{X}_{1,1} = (0,0,2)$, $\mathbf{X}_{2,2} = (0,0,2)$, and $\mathbf{X}_{1,2} = \mathbf{X}_{2,1} = (0,0,0)$. Therefore we have

$$e \ = \ \mathbf{N}\cdot\mathbf{X}_{1,1} \ = \ \alpha^{-1}\cdot 2 \ = \ g \ = \ \mathbf{N}\cdot\mathbf{X}_{2,2}\ , \quad f \ = \ \mathbf{N}\cdot\mathbf{X}_{1,2} \ = \ 0$$

so that the Second Fundamental Form is equal to

$$\frac{2(du\, du \ + \ dv\, dv)}{\alpha}\ .$$

The Gaussian curvature is then given by the quotient

$$\frac{eg-f^2}{EG-F^2}$$

where the preceding computations show that the numerator is $4/\alpha^2$ and the denominator is

$$(1+4u^2)(1+4v^2) \ - \ (4u^2v^2)^2 \ = \ 1+4u^2+4v^2 \ = \ \alpha^2$$

so that the Gaussian curvature K is equal to $4/\alpha^4$.

Since α is positive everywhere, the same is true for K.

6. [*25 points*] Let A be a 2×2 matrix, and let I and O denote the identity and zero matrices of the same size. Prove the identity $A^2 \; - \; \mathrm{trace}\,(A) \cdot A \; + \; \det A \cdot I \;\; = \;\; O$ (the Cayley-Hamilton Theorem).

SOLUTION

Let A be the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so that the trace and determinant of A are $ad - bc$ and $a + d$ respectively. Then we have

$$A^2 \quad = \quad \begin{pmatrix} a^2 + bc & ab + bd \\ c & d \end{pmatrix} \;, \qquad - \;\mathrm{trace}\,(A) \cdot A \quad = \quad - \begin{pmatrix} a^2 + ad & ab + bd \\ ac + dc & ad + d^2 \end{pmatrix}$$

$$\det A \cdot I \quad = \quad \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$$

and if we add these three matrices we obtain the zero matrix.