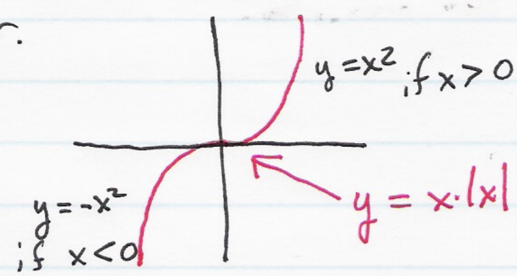
FINITELY DIFFERENTIABLE FUNCTIONS

In class I mentioned that for each positive integer n there are functions which have n continuous derivatives but do not have an $(n+1)$ st derivative. Here is a more detailed discussion.

Continuous derivative but no second derivative.

Let $f(x) = x|x|$. Its graph is two half-parabolas, each pointing in the opposite directions to each other.



One can check directly that $f'(x) = 2|x|$. This is done on a case by case basis, depending on whether $x > 0$, $x < 0$, or $x = 0$.

Now f has no second derivative because one cannot define $f''(0)$; $f''(x) = 2$ if $x > 0$ and $f''(x) = -2$ if $x < 0$.

Now let $f_n(x) = x^n|x|$ if $n \geq 0$.

CLAIM: $f_n'(x) = (n+1) f_{n-1}(x)$. if $n \geq 3$.

Derivation We have $f_n(x) = x^{n-1} f_1(x)$.

Apply the product differentiation rule:

$$\frac{d}{dx}\left(x^{n-1} f_1(x)\right) = (n-1)\, x^{n-2} f_1(x) + x^{n-1} f_1'(x) =$$

$$(n-1)\, x^{n-1}|x| + 2\, x^{n-1}|x| = (n+1)\, x^{n-1}|x| \blacksquare$$

The preceding lead to the identity

$$\frac{d^n}{dx^n} f_n(x) = n!\,|x|$$

which shows that $f_n(x)$ has a continuous nth order derivative. However, there is no $(n+1)$st derivative because $|x|$ has no derivative at zero.