Footnotes for Section IV.5

Here are some further comments on a few points and some corrections for misprints in the notes.

DOUBLY RULED QUADRIC SURFACES. In the notes we mentioned that certain quadric surfaces were examples of ruled surfaces with negative Gaussian curvature. In fact, there are two examples that are doubly ruled. Since some of the more accessible references contain mistakes in the formulas, we shall describe these descriptions of the quadrics as doubly ruled surfaces explicitly.

The one-sheeted hyperboloid defined by $x^2+y^2-z^2=1$ has the following separate descriptions as a ruled surface:

$$(\cos u, \sin u, 0) \ + \ v\,(-\sin u, \cos u, \varepsilon)$$

$$(\text{where } \varepsilon = \pm 1)$$

The hyperbolic paraboloid (saddle surface) defined by $z = x^2 - y^2$ has the following separate descriptions as a ruled surface:

$$(u, 0, u^2) \ + \ v\,(1.\varepsilon, 2u)$$

$$(\text{where } \varepsilon = \pm 1)$$

MISPRINTS. In the derivation of the Gaussian curvature formula for surfaces of revolution and the description of the tractrix on pages 94 – 97 there are a few small misprints.

page 94, *line* -5. The first coordinate should be zero.
page 94, *line* -11. The third coordinate should be $q(v)\sin v$.
page 95, *middle of page*. In the formulas for Gaussian curvature, the term $p''q$ should be replaced by $p''q'$ both times that the former appears.
page 96, *line* -5. The equation $|s\mathbf{c}'(t)| = a$ should read $|s\mathbf{D}'(t)| = a$.

COMPUTING THE GAUSSIAN CURVATURE OF THE PSEUDOSPHERE. Here are a few additional details; since the full computation is elementary but messy and not particularly enlightening, we shall merely explain how one writes the curvature in terms of trigonometric functions of θ. We have the parametric equations $x = p(\theta)$ and $y = q(\theta)$ for the tractrix as in the notes, where $q(\theta) = a\sin\theta$. The latter yields the equations $q'(\theta) = a\sin\theta$ and $q''(\theta) = -a\sin\theta$. We also have $x = p(\theta)$, where p is given on line 6 of page 97, but all we need to apply the formula fo K on page 95 are the first and second derivatives of p. Using the formula at the bottom of page 96, we obtain the following description of $p'(\theta)$:

$$\frac{dp}{d\theta} \ = \ \frac{\sqrt{a^2-q^2}}{q}\cdot\frac{dq}{d\theta} \ = \ \frac{a\cos^2\theta}{\sin\theta} \ = \ a\cos^2\theta\,\csc\theta$$

This leads to the following formula for the second derivative:

$$p'' \ = \ a\big(-2\cos^2\theta \ - \ \cos^3\theta\csc^2\theta\big)$$

It follows that K is equal to a quotient of two polynomials in the standard trigonometric functions times $1/a^2$. If one sits down and applies standard trigonometric identities to the expression obtained by substituting p', p'', q, q' and q'' into the (corrected) Gaussian curvature formula

$$K \ = \ \frac{(\,p''\,q' \ - \ p'\,q'')\,p'}{[(p')^2 \ + \ (q')^2]^2\,q}$$

then it turns out that the expression does simplify to $-1/a^2$ as claimed.■