V. Further Topics

This continuation of the course lecture notes discusses two fundamental topics in the classical theory of surfaces. The first is a basic result of Gauss which states that the Gaussian curvature of a surface depends only on the First Fundamental Form of the surface. This fact allows one to define curvature for a much broader range of geometric objects, an idea that has been basic to differential geometry for nearly two centuries. The second topic is an analog of the basic existence and uniqueness results for curves based upon the Frenet-Serret formulas. Just as curvature and torsion determine curves in 3-space, the First and Second Fundamental Forms provide a similarly complete characterization of surfaces.

V.1 : Compatibility equations, *Theorema Egregium*

(O'Neill, §§ 6.4–6.5)

One of the most far-reaching results on the differential geometry of surfaces is that the Gaussian curvature of a surface can be expressed entirely in terms of the First Fundamental Form:

GAUSS' THEOREMA EGREGIUM. *If* $\mathbf{X}$ *is a* $1-1$ *regular parametrization such that the First Fundamental Form is given by*

$$E(u,v)\,du\,du \;+\; 2\,F(u,v)\,du\,dv \;+\; G(u,v)\,dv\,dv$$

and $K(u,v)$ *is the Gaussian curvature function, then the Gaussian curvature depends only upon the coefficients of the First Fundamental Form of the surface and their partial derivatives.*

In contrast, the plane and cylinder have the same First Fundamental Form but different mean curvatures.

At the end of Section III.4 in the course lecture notes, we discussed generalizations of the First Fundamental Form known as Riemannian metrics. One can use the formula above to define Gaussian curvature with respect to an arbitrary Riemannian metric regardless of whether it comes from a First Fundamental Form. This is an important step in formulating general notion of curvature in differential geometry that can be used in many different contexts and have a dramatically wide range of applications in mathematics and physics.

Discussion of the proof

The results of this section and the next are based upon an analysis of the second partial derivatives of a regular parametrization $\mathbf{X}$. In some sense this is analogous to the idea behind the Frenet-Serret Formulas for curves; one writes out the various derivatives as linear combinations of simpler objects and looks for useful interrelationships. In the case of curves, the Frenet Trihedron provided a useful basis for $\mathbf{R}^3$ at each point of the curve. For surfaces given by regular parametrizations, the corresponding useful basis is given by the partial derivatives $\mathbf{X}_1$ and $\mathbf{X}_2$ together with the unit normal vector $\mathbf{N}$, which may be viewed as $\mathbf{X}_1 \times \mathbf{X}_2$ normalized to have unit length. One major difference with the theory for curves is that these bases are usually not orthonormal, but this turns out to be a relatively minor issue that can be addressed directly using linear algebra as in the final portion of Section IV.3 of the course lecture notes.

If one writes out the partial derivatives of $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{N}$ with respect to the u (first) and v (second) variables and uses the earlier computations involving the First and Second Fundamental Forms, one obtains the following sorts of formulas in which the quantities $\Gamma^i_{j,k}$ are smooth functions of u and v and are called *Christoffel symbols of the second kind*; the terminology is chosen to be consistent with concepts in tensor analysis (see the bottom of page 213 in the Schaum's Outline Series book on differential geometry for further information).

$$\mathbf{X}_{1,1} = \Gamma^1_{1,1}\,\mathbf{X}_1 + \Gamma^2_{1,1}\,\mathbf{X}_2 + e\,\mathbf{N}$$

$$\mathbf{X}_{1,2} = \Gamma^1_{1,2}\,\mathbf{X}_1 + \Gamma^2_{1,2}\,\mathbf{X}_2 + f\,\mathbf{N}$$

$$\mathbf{X}_{2,2} = \Gamma^1_{2,2}\,\mathbf{X}_1 + \Gamma^2_{2,2}\,\mathbf{X}_2 + g\,\mathbf{N}$$

$$\mathbf{N}_1 = \beta^1_1\,\mathbf{X}_1 + \beta^2_1\,\mathbf{X}_2$$

$$\mathbf{N}_2 = \beta^1_2\,\mathbf{X}_1 + \beta^2_2\,\mathbf{X}_2$$

It is convenient to define $\Gamma^i_{2,1} = \Gamma^i_{1,2}$ for $i = 1,\ 2$ so that $\Gamma^i_{j,k}$ is defined for $1 \leq i,\ j,\ k \leq 2$ and satisfies $\Gamma^i_{k,j} = \Gamma^i_{j,k}$.

Using the methods described in the last part of Section IV.3 in the course lecture notes, one can solve for β^i_j in terms of the coefficients of the First and Second Fundamental Forms:

$$\beta_{1,1} = \frac{f\,F - e\,G}{E\,G - F^2}$$

$$\beta_{2,1} = \frac{e\,F - f\,E}{E\,G - F^2}$$

$$\beta_{1,2} = \frac{f\,F - f\,G}{E\,G - F^2}$$

$$\beta_{1,2} = \frac{f\,F - g\,E}{E\,G - F^2}$$

If one substitutes these into the equations for $\mathbf{N}_1$ and $\mathbf{N}_2$ one obtains the *Weingarten equations*. Computing the Christoffel symbols is more difficult. The following formulas are derived in Problem 10.3 on page 216 of the Schaum's Outline Series review of differential geometry that was cited previously:

$$\Gamma^1_{1,1} = \frac{G\,E_1 - 2\,F\,F_1 + F\,E_2}{2\,(E\,G - F^2)}$$

$$\Gamma^1_{1,2} = \frac{G\,E_2 - F\,G_1}{2\,(E\,G - F^2)}$$

$$\Gamma^1_{2,2} = \frac{2\,G\,F_2 - G\,G_1 + F\,G_2}{2\,(E\,G - F^2)}$$

$$\Gamma^2_{1,1} = \frac{2\,E\,F_1 - E\,E_2 + F\,E_1}{2\,(E\,G - F^2)}$$

$$\Gamma^2_{1,2} \;=\; \frac{E\,G_1 \;-\; F\,E_2}{2\,(E\,G \;-\; F^2)}$$

$$\Gamma^2_{2,2} \;=\; \frac{E\,G_1 \;-\; 2\,F\,F_2 \;+\; F\,G_1}{2\,(E\,G \;-\; F^2)}$$

It is important to note that *the Christoffel symbols depend only upon the coefficients of the First Fundamental Form and their first partial derivatives.*

The most direct approach to proving Gauss' theorem about the Gaussian curvature is to continue by proving that

$$K\,(E\,G \;-\; F^2)^2 \;=\; [\mathbf{X}_{1,1},\,\mathbf{X}_1,\,\mathbf{X}_2\,]\cdot[\mathbf{X}_{2,2},\,\mathbf{X}_1,\,\mathbf{X}_2\,] \;-\; [\mathbf{X}_{1,2},\,\mathbf{X}_1,\,\mathbf{X}_2\,]^2\,.$$

This compuation is carried out in Problem 10.4 on page 217 of the Schaum's Outline Series on differential geometry, and equivalent statements involving differential forms are established in Chapter 6 of O'NEILL (see pages 280–281 in particular). Further computations using the same methods then yield the identity

$$K\,(E\,G \;-\; F^2)^2 \;=\; \left(F_{1,2} \;-\; \tfrac{1}{2}\,E_{2,2} \;-\; G_{1,1}\right) +$$

$$\begin{vmatrix} 0 & F_2 - \frac{1}{2}\,G_1 & \frac{1}{2}\,G_2 \\ \frac{1}{2}\,E_1 & E & F \\ F_1 - \frac{1}{2}\,E_2 & F & G \end{vmatrix} \;-\; \begin{vmatrix} 0 & \frac{1}{2}\,E_2 & \frac{1}{2}\,G_1 \\ \frac{1}{2}\,E_2 & E & F \\ \frac{1}{2}\,G_1 & F & G \end{vmatrix}$$

which implies that K depends only upon the coefficients of the First Fundamental Form and their partial derivatives. It is an elementary exercise in partial differentiation to show that this equation is equivalent to the one in the statement of Gauss' theorem that is given above.■

Another approach to deriving Gauss' theorem is given on pages 231–235 in Section 4–3 of DO CARMO and (using differential forms) in Chapter 6 of O'Neill. This alternate approach also has other implications, and it will be discussed in the next section.

Curvature and the First Fundamental Form

We have discussed the geometric significance of Gaussian curvature for a surface in $\mathbf{R}^3$ in terms of its First and Second Fundamental Forms. The *Theorema Egregium* provides a way of defining the Gaussian curvature entirely in terms of the First Fundamental Form, and consequently for riemannian metrics that are not necessarily realizable by surfaces in $\mathbf{R}^3$. One is therefore led to natural questions about interpreting the Gaussian curvature entirely in terms of metrical properties directly given the First Fundamental Form without using auxiliary objects such as normal lines or osculating circles. We shall describe one interpretation of positive and negative Gaussian curvature at a point entirely in metric terms; if the Gaussian curvature is equal to zero the situation is more complicated, but if the Gaussian curvature is identically zero then we shall give a similar interpretation.

Given a Riemannian metric

$$E(u,v)\,du\,du \;+\; 2\,F(u,v)\,du\,dv \;+\; G(u,v)\,dv\,dv$$

and a parametrized regular, piecewise smooth curve in a connected domain $U \subset \mathbf{R}^2$ on which the metric is defined, one can define the **length** of the curve by the formula

$$\int_a^b \sqrt{E(u,v)\,u'(t)^2 \;+\; 2\,F(u,v)\,u'(t)\,v'(t)v \;+\; G(u,v)\,v'(t)^2}\;dt$$

where the curve is defined on the interval $[a,b]$. The positivity condition on the coefficients E, F and G for a Riemannian metric imply that the expression inside the square root sign is always positive for regular smooth curves. One would like to define the *distance between two points* with respect to this metric as the greatest lower bound of the lengths of all regular piecewise smooth curves joining the points.

Two questions immediately arise. First of all, one needs to show that the lengths of curves joining two distinct points are bounded from below by a positive constant; in other words, if $\mathbf{p}$ and $\mathbf{q}$ are distinct points of a surface then it is not possible to find a sequence of piecewise smooth regular curves $\mathbf{y}_n$ joining them such that the length of $\mathbf{y}_n$ is less than $1/n$. Second, one would like to know if there is some curve for which the greatest lower bound is actually realized. Such a curve is called a *minimal geodesic*.

It is fairly easy to construct a somewhat artificial example where there is no curve of minimum length joining two points. Specifically, consider the surface given by removing the origin from the xy-plane. Then the greatest lower bound of the lengths of all piecewise smooth curves joining $(1,0,0)$ and $(-1,0,0)$ is equal to 2, which is the ordinary Euclidean distance, but there is no curve of length 2 joining these points that misses the origin. To see this, let $\mathbf{y}$ be a regular piecewise smooth curve joining the two points in question that is defined on $[a,b]$. Then there is some point $\xi \in (a,b)$ such that the first coordinate of $\mathbf{y}(\xi)$ is equal to some nonzero value, say c. It then follows by the Intermediate Value Theorem that the arc length of $\mathbf{y}$ must be greater than or equal to the length of the broken line curve joining $(1,0,0)$ to $(0,c,0)$ linearly and $(0,c,0)$ to $(-1,0,0)$ linearly. The length of this broken line curve is $2\sqrt{1+c^2}$, which is strictly greater than 2. Therefore there is no curve of shortest length joining the two points that lies completely inside the surface. One obvious feature of this example is that one can extend the given surface to a larger one (namely, the whole plane) in which there is a curve of minimum length joining the two points in question. In fact, one can construct examples for which one cannot add extra points to ensure that minimizing geodesics always exist, but such a construction would require a great deal of additional work. A natural candidate for a bad example is the graph of the function

$$f(x,y) \quad = \quad \frac{x\,y}{x^2 \;+\; y^2}$$

which is defined for $(x,y) \neq (0,0)$ and cannot be extended to a function that is continuous at $(0,0)$.

In contrast to the preceding paragraph, it turns out that one can always find curves of minimum length joining a given point $\mathbf{p}$ to another point $\mathbf{q}$ provided $\mathbf{q}$ is sufficiently close to $\mathbf{p}$, and this fact has important implications to showing the the lengths of curves joining two distinct points are bounded from below by a positive constant.

EXISTENCE OF SHORT GEODESICS. *Suppose we are given a riemannian metric* $\mathbf{M}$ *on a connected domain in* $\mathbf{R}^2$*, and let* $\mathbf{p} \in U$*. Then there is an* $r > 0$ *such that* $|\mathbf{q} - \mathbf{p}| < r$ *implies that* $\mathbf{p}$ *and* $\mathbf{q}$ *can be joined by a regular piecewise smooth curve of least length, and this curve is in fact a regular smooth curve that lies entirely in the open disk with center* $\mathbf{p}$ *and radius* r*.*

Furthermore, given any nonzero vector $\mathbf{v} \in \mathbf{R}^2$ *there is a unique regular smooth curve curve* Γ *defined on an open interval* $(-h,h)$ *containing* 0 *such that* $\Gamma(0) = \mathbf{p}$*,* $\Gamma'(0) = \mathbf{v}$ *and* Γ *defines a curve of minimum length joining* $\mathbf{p}$ *to* $\Gamma(t)$ *for all* t *in the given interval* $(-h,h)$*.*

Finally, if $\delta \in (0, r)$ and $L(\mathbf{q})$ denotes the length of the shortest curve joining $\mathbf{p}$ *to* $\mathbf{q}$*, then the minimum value $m(\delta)$ of $L(\mathbf{q})$ over the circle defined by $|\mathbf{q}| = \delta$ is positive.*■

The curves of least length in this result are called **minimizing geodesics**. It turns out that such curves are defined by second order differential equations, and this is the reason for the conclusion in the second paragraph.

COROLLARY. *If Σ is a surface and* $\mathbf{p}$ *and* $\mathbf{q}$ *are two points on σ that can be joined by a regular piecewise smooth curve on Σ, then the set of lengths for all such curves is bounded from below by a positive constant.*

Proof. To simplify the discussion we shall choose parametrizations for our regular piecewise smooth curves over some interval of the form $[0, a]$ such that $\Gamma(0) = \mathbf{p}$ and $\Gamma(a) = \mathbf{q}$. We need to find a positive lower bound for the length that does not depend upon the particular curve Γ.

Let $\mathbf{X}$ be a regular smooth parametrization for Σ at $\mathbf{p}$ that is 1–1, let $\mathbf{X}(\mathbf{p}_0) = \mathbf{p}$, and let $r > 0$ be as in the existence theorem stated above. There are two cases, depending upon whether the point $\mathbf{q} \in \Sigma$ has the form $\mathbf{X}(\mathbf{q}_0)$ for some $\mathbf{q}_0$ satisfying $|\mathbf{q}_0 - \mathbf{p}_0| < r$.

FIRST CASE. Suppose that $\mathbf{q}$ satisfies the condition in the preceding sentence, and let $s = |\mathbf{q}_0 - \mathbf{p}_0|$. If $\mathbf{y}$ is an arbitrary point of Σ having the form $\mathbf{X}(\mathbf{y}_0)$ for some $\mathbf{y}_0$ satisfying $|\mathbf{y}_0 - \mathbf{p}_0| < r$, then we shall define $g_0(\mathbf{y})$ to be equal to $|\mathbf{y}_0 - \mathbf{p}_0|$; the right hand side is well defined because the parametrization $\mathbf{X}$ is 1–1. This turns out to be a continuous function of $\mathbf{y}$. Likewise, if we define a real valued function g by setting $g(t) = \min\{s,\ g_0(\,\Gamma(t)\,)\,\}$ if $\Gamma(t)$ has the given special form, and $g(t) = s$ if $\Gamma(t)$ does not have this form, then g is continuous on the interval $[a, b]$ over which Γ is defined.

Since $g(a) = 0$ and $g(b) = s$, there must be a first parameter value t_0 such that $g(t_0) = s$. We claim that the image of the restricted curve $\Gamma|[0, t_0)$ lies in the image W of the disk of radius s centered at $\mathbf{p}_0$ under $\mathbf{X}$ (in fact a stronger statement is true but we shall not need this). This is true because if $\Gamma(t)$ does not lie in the image then $g(t) \geq s$ and we know that $g(t) < s$ if $t \in [0, t_0)$.

By the Intermediate Value Theorem there is a $t_1 \in (0, t_0)$ such that $g(t_1) = \frac{1}{2}s$; we know that the image of Γ restricted to $[0, t_1]$ lies in the set W described above, and therefore this restriction may be written as a composite $\mathbf{X} \circ \Gamma_1$ for some regular piecewise smooth curve Γ_1 which takes values in the disk of radius r centered at $\mathbf{p}_0$. We then have

$$\text{Length}\,(\Gamma|[0, t_1]) \quad = \quad \text{Length}_{\mathbf{M}}(\Gamma_1) \geq m(\tfrac{1}{2}\,s) \quad > \quad 0$$

on one hand and

$$\text{Length}\,(\Gamma|[0, t_1]) \quad \leq \quad \text{Length}(\Gamma)$$

on the other, which implies that the right had side is greater than or equal to the positive quantity $m(\frac{1}{2}s)$. This gives us our desired positive lower bound on the length of Γ which is independent of the curve Γ itself.

SECOND CASE. The argument is similar but not quite identical. We may define the function g exactly in the first case for an arbitrary s such that $0 < s < r$. In this case we know that $g(t) = s$ for some parameter value t because there is some value t such that $\Gamma(t)$ does **NOT** have the form $\mathbf{X}(\mathbf{y}_0)$ for some $\mathbf{y}_0$ satisfying $|\mathbf{y}_0 - \mathbf{p}_0| < r$. We can now proceed as before to find the least parameter value t_0 such that $g(t_0) = s$, and from this point on the argument is identical to the proof in the first case.■

FUNDAMENTAL PROPERTIES OF DISTANCE FUNCTIONS. *Suppose that we have either a Riemannian metric* $\mathbf{M}$ *defined on a connected domain U in* $\mathbf{R}^n$ *or a geometric surface Σ*

in $\mathbf{R}^3$ *such that each pair of points in* Σ *can be joined by a regular piecewise smooth curve in* Σ*, and let* $d_{\mathbf{M}}(\mathbf{x}, \mathbf{y})$ *or* $d_\Sigma(\mathbf{x}, \mathbf{y})$ *denote the greatest lower bound of the lengths of piecewise smooth curves joining* $\mathbf{x}$ *and* $\mathbf{y}$ *in* U *or* Σ*. Then this distance function* d *has the following basic properties:*

[1] *The distance* $d(\mathbf{x}, \mathbf{y})$ *is nonnegative, and it is equal to zero if and only if* $\mathbf{x} = \mathbf{y}$*.*

[2] *For all* $\mathbf{x}$ *and* $\mathbf{y}$ *we have* $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$*. and it is equal to zero if and only if* $\mathbf{x} = \mathbf{y}$*.*

[3] (TRIANGLE INEQUALITY) *For all* $\mathbf{x}$*,* $\mathbf{y}$ *and* $\mathbf{z}$ *we have*

$$d(\mathbf{x}, \mathbf{z}) \quad \leq \quad d(\mathbf{x}, \mathbf{y}) \;+\; d(\mathbf{y}, \mathbf{z})$$

Sketch of proofs. The first statement follows from the immediately preceding discussion. To prove the second, not that if Γ is a regular piecewise smooth curve defined on $[a, b]$ joining $\mathbf{x}$ to $\mathbf{y}$ then $\Gamma^*(t) = \Gamma(b - t)$ defines a similar curve on $[a - b, 0]$ joining $\mathbf{y}$ to $\mathbf{x}$. This implies that $d(\mathbf{y}, \mathbf{x}) \leq d(\mathbf{x}, \mathbf{y})$. Reversing the roles of $\mathbf{x}$ and $\mathbf{y}$ yields the reverse inequality $d(\mathbf{y}, \mathbf{x}) \geq d(\mathbf{x}, \mathbf{y})$, and therefore the two quantities must be equal. Finally, to prove the third statement, let $\varepsilon > 0$ and choose suitable curves Γ_1 and Γ_2 such that Γ_i is defined on $[0, a_i]$, with Γ_1 joining $\mathbf{x}$ to $\mathbf{y}$ and Γ_2 joining $\mathbf{y}$ to $\mathbf{z}$, and the lengths of these curves satisfying

$$\mathrm{Length}(\Gamma_1) \quad \leq \quad d(\mathbf{x}, \mathbf{y}) \;+\; \frac{\varepsilon}{2}$$

$$\mathrm{Length}(\Gamma_2) \quad \leq \quad d(\mathbf{y}, \mathbf{z}) \;+\; \frac{\varepsilon}{2}\,.$$

Consider the curve formed by concatenating Γ_1 and Γ_2; specifically, let Γ be the curve defined on the interval $[0, a_1 + a_2]$ such that $\Gamma(t) = \Gamma_1(t)$ for $t \in [0, a_1]$ and $\Gamma(t) = \Gamma(t - a_1)$ for $t \in [a_1, a_1 + a_2]$. These piece together to form a regular piecewise smooth curve because the two formulas yield the same point at parameter value a_1. The length of this curve then given by

$$\mathrm{Length}(\Gamma_1) \;+\; \mathrm{Length}(\Gamma_2)$$

and hence we have the inequality

$$d((\mathbf{x}, \mathbf{z}) \quad = \quad \mathrm{Length}(\Gamma) \quad = \quad \mathrm{Length}(\Gamma_1) \;+\; \mathrm{Length}(\Gamma_2) \quad \leq \quad d(\mathbf{x}, \mathbf{y}) \;+\; d(\mathbf{y}, \mathbf{z}) \;+\; \varepsilon$$

for every $\varepsilon > 0$. In particular this implies that the expression on the left hand side cannot be greater than $d(\mathbf{x}, \mathbf{y}) \;+\; d(\mathbf{y}, \mathbf{z})$, and this is precisely the assertion in [3].■

A METRIC INTERPRETATION OF CURVATURE. Suppose that we are given three points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in $\mathbf{R}^3$ that form the vertices of an isosceles triangle with vertex at $\mathbf{a}$; *i.e.*, we have $|\mathbf{b} - \mathbf{a}| = |\mathbf{c} - \mathbf{a}| = \ell > 0$. If θ is the angle between $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$ then it is an elementary exercise in trigonometry to prove that $|\mathbf{c} - \mathbf{b}| = 2 \sin \frac{1}{2}\theta$. Roughly speaking, Gaussian curvature measures the extent to which this fails for riemannian metrics. The proof of this fact requires a considerable amount of machinery from Riemannian geometry, so we shall simply state the results here.

Since we are only concerned with metric behavior near a point, it will suffice to look at Riemannian metrics defined on an open disk centered at some point $\mathbf{p}$ in a connected domain $U \subset \mathbf{R}^2$. Let $\mathbf{M}$ be a Riemannian metric, and let $r > 0$ be so small that every point in the open disk of radius r centered at $\mathbf{p}$ can be joined to the later by a smooth curve of minimum length lying entirely inside this disk. Given two linearly independent vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{R}^2$, let $\theta_{\mathbf{M}}(\mathbf{p})$ be the angle between them computed with respect to the Riemannian metric:

$$\cos\left(\theta_{\mathbf{M}}(\mathbf{p})\right) \quad = \quad \frac{\mathbf{M}_{\mathbf{p}}(\mathbf{v}, \mathbf{w})}{\left(\mathbf{M}_{\mathbf{p}}(\mathbf{v}, \mathbf{v})\right)^{1/2}\;\left(\mathbf{M}_{\mathbf{p}}(\mathbf{w}, \mathbf{w})\right)^{1/2}}$$

Consider now the smooth geodesics which pass through $\mathbf{p}$ and have tangent vectors $\mathbf{v}$ and $\mathbf{w}$ at $\mathbf{p}$. We can find points on these geodesics that are some positive distances away from $\mathbf{p}$; if δ_0 is the minimum of the two distances, then for every $\ell \in (0, \delta_0]$ we can find points $\mathbf{x}$ and $\mathbf{y}$ on the respective geodesics such that the distances from $\mathbf{x}$ and $\mathbf{y}$ to $\mathbf{p}$ are both equal to ℓ (suppose that we have geodesics with the given tangent vectors defined on intervals $[0, a]$ and $[0, b]$ respectively; then by the Intermediate Value Theorem one can find points s_0 and t_0 in these intervals so that the lengths of the restrictions of the geodesics up to parameter values s_0 and t_0 are equal to ℓ). We then have the following relationships between the Gaussian curvature at $\mathbf{p}$ and the distance between $\mathbf{x}$ and $\mathbf{y}$ with respect to $\mathbf{M}$.

DISTANCE COMPARISON. *Suppose we are given everything as in the preceding discussion, and let K be the Gaussian curvature at $\mathbf{p}$.*

(*i*) *If $K > 0$ then there is a $\delta_1 > 0$ such that if $\ell < \delta_1$ we have*

$$d_{\mathbf{M}}(\mathbf{x},\, \mathbf{y}) \quad < \quad 2\,\ell\, \sin \theta_{\mathbf{M}}(\mathbf{p}) \ .$$

(*ii*) *If $K < 0$ then there is a $\delta_1 > 0$ such that if $\ell < \delta_1$ we have*

$$d_{\mathbf{M}}(\mathbf{x},\, \mathbf{y}) \quad > \quad 2\,\ell\, \sin \theta_{\mathbf{M}}(\mathbf{p}) \ .$$

(*iii*) *If the Gaussian curvature is identically zero, then there is a $\delta_1 > 0$ such that if $\ell < \delta_1$ we have*

$$d_{\mathbf{M}}(\mathbf{x},\, \mathbf{y}) \quad = \quad 2\,\ell\, \sin \theta_{\mathbf{M}}(\mathbf{p}) \ . \blacksquare$$

These may be viewed as generalizations of standard trigonometric formulas from spherical geometry, Noneuclidean geometry in the sense of Bólyai and Lobachevsky, and classical Euclidean geometry respectively. Note that if we only know the Gaussian curvature is zero at $\mathbf{p}$ but we know nothing else about its behavior near $\mathbf{p}$, then these comparison results yield no information.

V.2 : Fundamental Theorem of Local Surface Theory

(do Carmo, §§6.1–6.2, 6.5, 6.9)

The Frenet-Serret Formulas imply that curvature and torsion completely determine a curve locally provided on gives the initial position and unit tangent vector for the curve. There is a corresponding theorem for surfaces involving the coefficients E, F, G and e, f, g of the First and Second Fundamental Forms. However, these coefficient functions must satisfy some nontrivial restrictions. We have already noted that the matrix for the First Fundamental Form

$$\begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix}$$

must have positive eigenvalues, or equivalently that E and G as well as the determinant $E\,G - F^2$ must be positive. However, there are also other conditions that arise naturally from our basic assumptions that a local parametrization $\mathbf{X}$ have "sufficiently many" continuous partial derivatives. In particular, if we want $\mathbf{X}$ to have continuous third partial derivatives then we have equations of the form $\mathbf{X}_{1,1,2} = \mathbf{X}_{1,2,1} = \mathbf{X}_{2,1,1}$ and then we have equations of the form $\mathbf{X}_{2,2,1} = \mathbf{X}_{2,1,2} = \mathbf{X}_{1,2,2}$.

If we combine these equations with the expansions of the second partial derivatives $\mathbf{X}_{i,j}$ in terms of Christoffel symbols and the Second Fundamental Form coefficients, we obtain the following three equations:

$$e_2 - f_1 \;=\; e\,\Gamma^1_{1,2} \;+\; f\,\left(\Gamma^2_{1,2} \;-\; \Gamma^1_{1,1}\right) \;-\; g\,\Gamma^2_{1,1}$$

$$f_2 - g_1 \;=\; e\,\Gamma^1_{2,2} \;+\; f\,\left(\Gamma^2_{2,2} \;-\; \Gamma^1_{1,2}\right) \;-\; g\,\Gamma^2_{1,2}$$

$$e\,g \;-\; f^2 \;=\; F\cdot\left[(\Gamma^2_{2,2})_1 \;-\; (\Gamma^2_{1,2})_2 \;+\; \Gamma^1_{2,2}\Gamma^2_{1,1} \;-\; \Gamma^1_{1,2}\Gamma^2_{1,1}\right] \;+$$

$$E\cdot\left[(\Gamma^1_{2,2})_1 \;-\; (\Gamma^1_{1,2})_2 \;+\; \Gamma^1_{2,2}\Gamma^1_{1,1} \;+\; \Gamma^2_{2,2}\Gamma^1_{1,2} \;-\; \Gamma^1_{1,2}\Gamma^1_{1,2} \;-\; \Gamma^2_{1,2}\Gamma^1_{2,2}\right]$$

The first two of these are known as the *Codazzi-Mainardi Equations*. We note in passing that the third equation provides another demonstration of Gauss' Theorema Egregium; in fact, one important advantage of this proof is that it reflects the standard approach to curvature in the study of differential geometry for objects whose dimensions are greater than two.

The verifications of these formulas from the classical viewpoint are carried out on pages 235–236 of DO CARMO and in Problem 10.28 on page 224 of the Schaum's Outline Series book on differential geometry. Derivation of the corresponding formulas involving differential forms are given on pages 257, 260 and 281 of O'NEILL (see Theorem 1.7, Corollary 2.3 and Theorem 5.4 respectively).■

The Gauss and Codazzi-Mainardi equations play an important role in establishing the main result of this section.

FUNDAMENTAL THEOREM OF LOCAL SURFACE THEORY. *Let U be a connected domain in $\mathbf{R}^2$, and let E, F, G and e, f, g be smooth functions with sufficiently many continuous partial derivatives on U such that E, F and G satisfy the positive definiteness conditions given above and e, f and g satisfy the three compatibility conditions displayed above. Then for each $\mathbf{p}_0 \in U$, $\mathbf{p} \in \mathbf{R}^3$ and plane Π containing $\mathbf{p}$, there is a regular surface parametrization $\mathbf{X}$ defined on some open disk N about $\mathbf{p}_0$ such that the First and Second Fundamental Forms of $\mathbf{X}$ have coefficients equal to E, F, G and e, f, g respectively. This parametrization is locally unique up to a rigid motion of $\mathbf{R}^3$.*

The uniqueness proof is essentially a relatively lengthy argument involving the uniqueness of solutions of certain ordinary differential equations (see pages 236 and 311–314 of DO CARMO or the argument following the statement of Theorem 10.4 on pages 203–204 of the Schaum's Outline Series book on differential geometry). On the other hand, the existence proof requires the solution of a system of partial differential equations.

In order to prove the existence of a regular smooth surface parametrization it is necessary to solve partial differential equations of the form $D\,y = A(\,x, y\,)$ where x and y are vectors and A is a smooth matrix valued function of x and y. In contrast to the situation for ordinary differential equations, the partial differential equation given above does not necessarily have a solution; specifically, the standard mixed partial derivative identities

$$\frac{\partial^2}{\partial x_i\,\partial x_j} \;=\; \frac{\partial^2}{\partial x_j\,\partial x_i}$$

imply that the entries of $A(x, y)$ and their partial derivatives must satisfy certain equations. However, the following result of F. G. Frobenius ensures that solutions always exist provided these conditions are satisfied:

FROBENIUS INTEGRABILITY THEOREM. *Let $n = k + d$, identify $\mathbf{R}^n$ with $\mathbf{R}^k \times \mathbf{R}^d$, let U be a connected domain in $\mathbf{R}^n$ and Let $\mathbf{A}$ be a smooth function defined on U and taking values in the space of $d \times k$ matrices, and let $(\mathbf{a}.\,\mathbf{b}\,) \in U$. Denote the entries of $\mathbf{A}$ by $A_{i,j}$.*

Assume in addition that these functions satisfy the compatibility conditions

$$\frac{\partial A_{i,j}}{\partial x_r} + \sum_{s=1}^{d} \frac{\partial A_{i,j}}{\partial x_s} A_{s,r} = \frac{\partial A_{i,r}}{\partial x_j} + \sum_{s=1}^{d} \frac{\partial A_{i,r}}{\partial x_s} A_{s,j} .$$

Then there exists a unique function Φ *defined on an open disk* V *containing* $\mathbf{a}$ *and taking values in* $\mathbf{R}^d$ *such that the following conditions hold:*

[1] $\Phi(\mathbf{a}) = \mathbf{b}$

[2] $(\mathbf{x}, \Phi(\mathbf{x})) \in U$ *for all* $\mathbf{x} \in V$.

[3] $D\,\Phi(\mathbf{x}) = \mathbf{A}(\mathbf{x}, \Phi(\mathbf{x}))$

Conversely, if such a function exists then the compatibility condition is satisfied.

Biographical information on Frobenius, and also many other mathematicians, may be found at the following online site:

`http://www-gap.cds.st-and.ac.uk/~history/BiogIndex.html`

The proof of the existence portion of the Fundamental Theorem of Local Surface Theory is discussed on pages 311-314 of DO CARMO as well as in Appendix 2 on pages 264–265 of the Schaum's Outline Series book on differential geometry.■

Final remarks

1. A discussion of the Fundamental Theorem of Local Surface Theory from the viewpoint in Chapter 6 of O'NEILL appears in Section 6.9 of the latter, with the main result appearing as Theorem 9.2 on pages 306–307. As noted at the end of this discussion, the result is completely analogous to the congruence theorem for curves discussed in Section II.4 of the course lecture notes.

2. A generalization of the Fundamental Theorem of Local Surface Theory to hypersurfaces of dimension $(n-1)$ in $\mathbf{R}^n$ is established in Section 9.2 of Hicks, *Notes on Differential Geometry*; the argument is a direct generalization of the proof for surfaces.■