Gram matrices and their determinants

V is a 2-dimensional inner product space with inner product $\langle , \rangle$. Suppose that $\{x, y\}$ is a basis for V. Then the Gram matrix of $\langle , \rangle$ with respect to $\{x, y\}$ is equal to

$$G = \begin{pmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle x, y \rangle & \langle y, y \rangle \end{pmatrix}.$$

CLAIM The determinant of this matrix is positive.

The derivation uses the Cauchy-Schwarz inequality, which is true in every inner product space:

$$\boxed{|\langle v, w\rangle| \le |v|\cdot|w| \text{ with equality } \Longleftrightarrow v \text{ and } w \text{ are linearly dependent}}$$

To prove the claim, first write out det G:

$$\det G = \langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2 = |x|^2|y|^2 - \langle x, y\rangle^2.$$

Since $\{x, y\}$ is a basis, the Cauchy Schwarz inequality implies that $\det G > 0$.

<u>Note</u>: A corresponding result for Gram matrices on higher dimensional inner product spaces is part of the Principal Minors Theorem on page 84 of the following document:

$\boxed{\text{http://math.ucr.edu/~res/linalgnotes.pdf}}$