The helix (or spiral) curve

This is a discussion of an example worked out in class. The simple helix or spiral is given by the following parametric form:

$$\mathbf{x}(t) \quad = \quad (\cos t, \sin t, t)$$

It follows that $\mathbf{x}'(t) = (-\sin t, \cos t, 1)$, so that the length is given by $|\mathbf{x}'(t)| = \sqrt{2} = s'(t)$ and consequently we may take the arc length parametrization to be give by $s = \sqrt{2}\,t$ or $t = s/\sqrt{2}$. One then has the following formula for the unit tangent vector:

$$\mathbf{T}(s) \quad = \quad \frac{1}{\sqrt{2}} \left(-\sin\left[\frac{s}{\sqrt{2}}\right], \cos\left[\frac{s}{\sqrt{2}}\right], 1 \right)$$

It follows that the curvature is equal to $|\mathbf{T}'| = \frac{1}{2}$ and the principal normal is given as follows:

$$\mathbf{N}(s) \quad = \quad \left(-\cos\left[\frac{s}{\sqrt{2}}\right], -\sin\left[\frac{s}{\sqrt{2}}\right], 0 \right)$$

If we compute $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$ we obtain the following formula:

$$\mathbf{B}(s) \quad = \quad \frac{1}{\sqrt{2}} \left(\sin\left[\frac{s}{\sqrt{2}}\right], -\cos\left[\frac{s}{\sqrt{2}}\right], 1 \right)$$

It follows that $\mathbf{B}'(s) = -\frac{1}{2}\,\mathbf{N}(s)$, so that the torsion is equal to $\frac{1}{2}$ at all points. Therefore the helix is an example of a curve with constant curvature and torstion.

If one changes the radius of the circle and takes the third coordinate to be an arbitrary nonzero number, then one still obtains curves with the constant curvatures and (nonzero) torsions. It is an instructive exercise to see how these constant values vary as one varies the radius and the vertical component of the curve.

The Fundamental Theorem of Local Curve Theory essentially implies the following characterization of helix curves:

PROPOSITION. *If a regular smooth curve in $\mathbb{R}^3$ has constant positive curvature and constant nonzero torsion, then this curve is congruent to a helix.*

If the Frenet trihedron of the curve at some initial point is the same as that for the helix, then this follows from the results of Unit I; the statement about congruence requires material from Section II.4.■