The helix (or spiral) curve – II

In the first document on this example we mentioned the question of determining what happens to the curvature and torsion if we change the radius of the circle or the rate at which the third coordinate is increasing. Here is the general situation:

Consider the modified helix with parametrization

$$\mathbf{x}(t) \quad = \quad (a\cos t, a\sin t, bt) \ .$$

It follows that $\mathbf{x}'(t) = (-a\sin t, a\cos t, b)$, so that the length is given by $|\mathbf{x}'(t)| = \sqrt{a^2+b^2} = s'(t)$ and consequently we may take the arc length parametrization to be give by $s = \sqrt{a^2+b^2}\,t$ or $t = s/\sqrt{a^2+b^2}$. One then has the following formula for the unit tangent vector:

$$\mathbf{T}(s) \quad = \quad \frac{1}{\sqrt{a^2+b^2}} \left(-a\sin\left[\frac{s}{\sqrt{a^2+b^2}}\right], a\cos\left[\frac{s}{\sqrt{a^2+b^2}}\right], b \right)$$

This in turn yields the following formula for the derivative of the unit tangent vector with respect to arc length:

$$\mathbf{T}'(s) \quad = \quad \frac{1}{a^2+b^2} \left(-a\cos\left[\frac{s}{\sqrt{a^2+b^2}}\right], -a\sin\left[\frac{s}{\sqrt{a^2+b^2}}\right], 0 \right)$$

It follows that the curvature is equal to $|\mathbf{T}'| = \frac{a}{a^2+b^2}$ and the principal normal is given as follows:

$$\mathbf{N}(s) \quad = \quad \left(-\cos\left[\frac{s}{\sqrt{a^2+b^2}}\right], -\sin\left[\frac{s}{\sqrt{a^2+b^2}}\right], 0 \right)$$

If we compute $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$ we obtain the following formula:

$$\mathbf{B}(s) \quad = \quad \frac{1}{\sqrt{a^2+b^2}} \left(b\sin\left[\frac{s}{\sqrt{a^2+b^2}}\right], -b\cos\left[\frac{s}{\sqrt{a^2+b^2}}\right], a \right)$$

It follows that $\mathbf{B}'(s) = -\frac{b}{a^2+b^2}\,\mathbf{N}(s)$, so that the torsion is equal to $\frac{b}{a^2+b^2}$ at all points. Therefore the modified helix is also an example of a curve with constant curvature and torstion.