Riemannian metrics and hyperbolic geometry: II

Gauss' *Theorema Egregium*, which is discussed in the document `furthertopics.*`, yields a very simple but fundamental property of the non-Euclidean plane we considered in Section III.4 of the lecture notes and the document `hyperbolic1.*`. The starting point is the fact that one can define the Gaussian curvature of a surface entirely in terms of its fundamental form and hence one can define Gaussian curvature for an arbitrary riemannian metric on a connected domain in $\mathbf{R}^2$. The following observation is an immediate consequence of the definition of Gaussian curvature entirely in terms of the First Fundamental Form:

PROPOSITION. *Let U be a connected domain in $\mathbf{R}^2$, and let g be a riemannian metric on U. Suppose that $f : U \to U$ is a riemannian isometry and $\mathbf{p} \in U$. Then the Gaussian curvature at $\mathbf{p}$ is equal to the Gaussian curvature at $f(\mathbf{p})$.*∎

This has an immediate implication for the Poincaré metric.

COROLLARY. *The Poincaré metric has constant Gaussian curvature.*

Proof. This is true because for each pair of points $\mathbf{p}$ and $\mathbf{q}$ there is a riemannian isometry taking $\mathbf{p}$ to $\mathbf{q}$. Therefore the Gaussian curvatures at these two points are equal, and since these points are arbitrary it follows that the Gaussian curvature is the same at every point.∎

It turns out that the Gaussian curvature for the Poincaré metric we have defined on the open unit disk is equal to -4; this follows from the methods used in Corollary 2.3 and Example 2.6 on pages 319 and 320 of O'NEILL. Thus the open unit disk with the Poincaré metric may be viewed as an analog of the standard metrics on the Euclidean plane and sphere of radius r, which have Gaussian curvatures 0 and $1/r^2$ respectively. If we multiply the Poincaré metric by a positive constant c, then it will follow that the Gaussian curvature of the new metric is equal to $-4/c^2$ (this also follows from the references to O'NEILL given aboe), so it the Poincaré metric and its positive multiples can have negative Gaussian curvature equal to an arbitrary negative real number.

Surfaces with constant Gaussian curvature, and their higher dimensional generalizations, play a fundamental role in differential geometry. Additional discussion of this topic appears in Section 8.6 of O'NEILL.