Power series and inverse functions

In the section of the notes on the Inverse Function Theorem (Section II.3), there was an assertion that if a function $y = f(x)$ has a convergent power series expansion at $x = a$ and $f'(a) \neq 0$, then the inverse function $x = g(y)$ has a convergent power series expansion at $y = b = f(a)$. The purpose of this document is to include some additional information about this; the statements of the results only require concepts from first year calculus, but the discussion of the proofs is more advanced and requires material from the theory of functions of a complex variable (Mathematics 165A–B).

The main results

The first 21 pages of the following online document provide a fairly complete summary of basic facts about infinite power series of the form

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

and we shall use the results stated in that (part of the) document:

`http://www.grossmount.edu/carylee/Ma280/PowerPoint/Power%20Series.ppt`

We shall also need the following result:

THEOREM. *Suppose that we are given functions $f(x)$ and $g(x)$ such that $f(0) = g(0) = 0$ and both have convergent power series in intervals $(-A, A)$ and $(-B, B)$. Then the composite function $h(x) = g\big(f(x)\big)$ also has a convergent power series representation on some interval $(-C, C)$.*

The conditions $f(0) = g(0) = 0$ were added for the sake of computational simplicity. The result holds more generally if f has a power series representation on $(a - A, a + A)$ and g has a power series representation on $\big(f(a) - B, f(a) + B\big)$, and the conclusion is that h has a power series representation on some interval of the form $\big(h(a) - C, h(a) + C\big)$.

In the theorem we can find the power series coefficients for h by several methods. For example, we can perform a direct substitution using $f(x) = \sum p_k x^k$ and $g(x) = \sum q_k x^k$, obtaining a messy looking expression of the form

$$\sum_k p_k \left(\sum_m q_m x^m \right)^k .$$

If we group together all terms in this expression which are constants times x^n for some n, we obtain something of the form $\sum_n r_n x^n$, and this turns out to be the power series expansion for h that we want. Alternatively, we know that the coefficients for this series are expressible in terms of the higher order derivatives of the composite function h, and we can use standard calculus identities to express the latter in terms of the higher order derivatives of f and g; this process will also yield the power series expansion for h.

We also need the following fact (not in the PowerPoint document explicitly):

UNIQUENESS OF POWER SERIES EXPRESSIONS. *If $f(x)$ is represented by two power series expressions over the same interval, then the corresponding coefficients of these series are all equal.*

Application to inverse functions

THEOREM. *Suppose now that f and g are inverse functions with $f(0) = g(0) = 0$ and $f'(0) \neq 0$, and suppose that $f(x)$ is given by a convergent power series over some interval $(-A, A)$. Then there is an interval $(-B, B)$ such that $B \leq A$ and $g(x)$ has a convergent power series expansion over the interval $(-B, B)$.*

If $f(x) = \sum\ p_k\, x^k$ and $g(x) = \sum\ q_k\, x^k$, then as above one can solve for the q_k in terms of the p_k by comparing coefficients in the expression

$$x \;=\; \sum_k\, p_k \left(\sum_m\, q_m\, x^m\right)^k \;=\; \sum_n\, r_n\, x^n$$

where as before the third expression is obtained from the second by combining like terms; in particular, by uniqueness of power series representations we must have $r_1 = 1$ and $r_n = 0$ otherwise. Note that $p_0 = q_0 = 0$ and $p_1 \neq 0$ for the examples we are considering.

There is also an identity called the *Lagrange Inversion Formula* that can be applied to find the coefficients q_k; here is an online site which discusses this formula:

`http://en.wikipedia.org/wiki/Lagrange_inversion_theorem`

One difficulty with the power series for the inverse function is that the interval of convergence $(-B, B)$ is often considerably smaller than one might expect. In particular, the functions $x + e^x - 2$ and $x^5 + x^3 + x$ satisfy the conditions of the theorem, the functions have positive derivatives everywhere, they have power series expansions at 0 which are valid for all x, and their limits as $x \to \pm\infty$ are equal to $\pm\infty$ — conditions which imply that the inverse functions can be defined for all values of x. However, it turns out that the power series expansions for the inverse functions are only valid on a **bounded interval** $(-B, B)$ and not for all real values of x. This is a consequence of the following result:

THEOREM X. *Suppose that $f(x)$ is a function which satisfies $f(0) = 0$, $f'(x) > 0$ everywhere, $f(x)$ has a convergent power series expansion which is valid for all real values of x, and*

$$\lim_{x \to \pm\infty}\ f(x) \;=\; \pm\infty\ .$$

Let g be the inverse function to x, and suppose that $g(x)$ also has a convergent power series expansion which is valid for all real values of x. Then $f(x) = kx$ for some positive constant k.

The proof of this result requires a number of concepts from the theory of functions of a complex variable and is beyond the scope of an elementary differential geometry course. However, since the functions $x + e^x - 2$ and $x^5 + x^3 + x$ satisfy the conditions of the theorem, it follows that the convergent power series expansion for the inverse function $g(x)$ CANNOT be valid for all real values of x and hence the interval of convergence must be bounded.

Proof of Theorem X

Suppose that we are given $f(x)$ and $g(x)$ as above, and assume further that $g(x)$ also has a power series expansion which is valid for all x. We need to show that $f(x)$ and $g(x)$ must be first degree polynomials.

First of all, basic results about analytic functions show that if $f(x)$ has a power series which converges for all real values of x, then the power series $f(z)$ also converges for all COMPLEX number z; of course the same holds for $g(z)$. Since f and g are inverses of each other and $f(0) = 0 = g(0)$ it follows that $f(g(x)) = x = g(f(x))$ for all real x in some small interval of the form $(-h, h)$. Since the points where two nonconstant analytic functions agree are isolated from each other, the observation in the previous sentence implies that we must have $f(g(z)) = z = g(f(z))$ for all complex z. In other words, it follows that *the functions f and g are inverse to each other as entire analytic functions.*

By the preceding paragraph, the proof of Theorem X reduces to verifying the following result:

LEMMA. *Let f be an entire analytic function which has an entire analytic inverse. Then f is a first degree polynomial.*

Proof of the Lemma. It will suffice to prove the result in the case where $f(0) = 0$, for if g is an arbitrary function satisfying the conditions in the lemma, then $f(z) = g(z) - g(0)$ still satisfies the conditions in the lemma and it also satisfies $f(0) = 0$. Therefore, if the lemma is true in the special case we can conclude that $f(z) = kz$ where $k \neq 0$. Since $g(z) = f(z) + g(0)$, this means that $g(z) = kz + g(0)$, where $k \neq 0$, and hence g also satisfies the condition in the conclusion of the lemma.

Suppose that the Taylor series expansion for f at 0 is given by $f(z) = \sum \; p_k \, z^k$ (this implies $p_0 = 0$). We need to show that $p_k = 0$ for all $k \geq 2$. There are two cases, depending upon whether or not infinitely many of these coefficients p_k are nonzero.

Suppose first that only finitely many p_k are nonzero, so that $f(z)$ is a polynomial of degree d for some $d \geq 1$. If $d \geq 2$, then $f(z)$ is a product of a nonzero constant and d linear polynomials $z - b_j$, where some of the roots b_j might be repeated. If there are at least two distinct roots, then $f(z) = 0$ for at least two values of z and hence f cannot have an inverse because it is not 1–1. Therefore there is only one root at the polynomial must have the form kz^d for some $k \neq 0$ and $d \geq 2$ (we know that 0 is a root because $f(0) = 0$). Since kz^d is not 1–1 if $d \geq 2$ it follows that d must be equal to 1. This proves the lemma if $f(z)$ is a polynomial.

Now suppose that infinitely many coefficients p_k in the Taylor series expansion of $f(z)$ at 0 are nonzero. Let $h(z) = f(1/z)$, so that $h(z)$ is an analytic function defined on $\mathbb{C} - \{0\}$. If the Taylor series expansion for f at 0 is given by $f(z) = \sum \; p_k \, z^k$, then the corresponding Laurent expansion for $h(z)$ is $h(x) = \sum \; p_k \, z^{-k}$. This means that h has an essential singularity at 0. By the Weierstrass-Casorati Theorem, it follows that for every complex number b there is a sequence of points $\{z_n\}$ such that $z_n \to 0$ and $\lim_{n\to\infty} \; f(z_n) = b$. Thus if $b \neq 0$ it also follows that $\lim_{n\to\infty} \; h(z_n) = 1/b$.

On the other hand, since f has an inverse, it follows that $\lim_{|z|\to\infty} \; |f(z)| = \infty$ and hence we also have $\lim_{z\to 0} \; |h(z)| = \infty$. This contradicts the final sentence of the previous paragraph, and hence it follows that the Taylor series expansion for $f(z)$ at 0 cannot have infinitely many nonzero terms.