Best linear approximations to curves

Given $\gamma(t)$ say $\gamma(0)=\vec{0}$

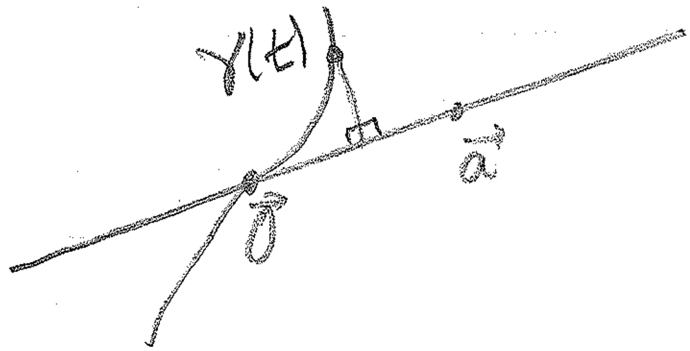
and γ has cont. 2nd derivative. $\gamma' \neq 0$ always.

Taylor's Formula: If t close to 0,

then $\gamma(t) = \underbrace{\gamma(0)}_{\vec{0}} + t\,\gamma'(0) + t^2\,\theta(t)$

where $\cancel{\lim_{t \to 0} \theta(t) = \vec{0}}$. $|\theta(t)| \le K$. const.

Let $\vec{a}$ be a unit vector. $g(t)$ = distance from $\gamma(t)$ to line through $\vec{0}$ and $\vec{a}$.



Vector algebra. $\perp$ proj $P_{\vec{a}}(\vec{v}) = (\vec{v} \cdot \vec{a})\,\vec{a}$ since $|\vec{a}| = 1$. Also

$Q_{\vec{a}}(\vec{v}) = \vec{v} - P_a(\vec{v})$ is $\perp$ $\vec{a}$.

$Q_{\vec{a}}(\vec{v}) = \vec{0} \iff \vec{v}$ is a multiple of $\vec{a}$.

Now $g(t) = \left| Q_{\vec{a}}(\gamma(t)) \right|$ and

$$Q_{\vec{a}}(\gamma(t)) = \gamma(t) - (\gamma(t)\cdot\vec{a})\,\vec{a}$$

$$t\gamma'(0) + t^2\theta(t) - t(\gamma'(0)\cdot\vec{a})\vec{a} -$$

$$t^2(\theta(t)\cdot\vec{a})\,\vec{a} = t\,Q_{\vec{a}}(\gamma'(0)) + t^2 Q_{\vec{a}}(\theta(t)).$$

Want $\vec{a}$ so that

$$g(t) \le t^2 L \quad L \text{ some constant.}$$

Note that

$$g(t) = |t| \left| Q_{\vec{a}}(\gamma'(0)) + t\,Q_{\vec{a}}(\theta(t)) \right|$$

In order to pull out another factor of $|t|$, we need $Q_{\vec{a}}(\gamma'(0)) = \vec{0}$, or equivalently $\gamma'(0)$ is a multiple of $\vec{a}$.

What if $Q_{\vec{a}}(\gamma'(0)) \neq 0$?

Use the inequality
$$|\vec{p}+\vec{q}| \geq |\vec{p}| - |\vec{q}|.$$

(follows from $|\vec{q}| + |\vec{p}+\vec{q}| = |-\vec{q}| + |\vec{p}+\vec{q}|$
$\geq |\vec{p} = (\vec{p}+\vec{q}) - \vec{q}|$).

Hence $g(t) \geq |t| \left(|Q_{\vec{a}}\gamma'(0)| - |t| \, Q_{\vec{a}}\theta(t) \right)$

so $\frac{g(t)}{t^2} \geq \frac{|Q_{\vec{a}}(\gamma'(0))|}{|t|} - |Q_a \theta(t)|$
$(t \neq 0)$

$|Q_{\vec{a}}(\vec{w})| \leq |\vec{w}|$

$$\text{RHS} \geq \frac{|Q_{\vec{a}}\gamma'(0)|}{|t|} - |\theta(t)| \geq$$

$$\frac{|Q_{\vec{a}}(\gamma'(0))|}{|t|} - K \quad \text{so} \quad \frac{g(t)}{t^2} \to \infty \text{ as}$$

$t \to 0$; hence $g(t) \not\leq t^2 \cdot$ constant.
if $Q_{\vec{a}}(\gamma'(0)) \neq \vec{0}$