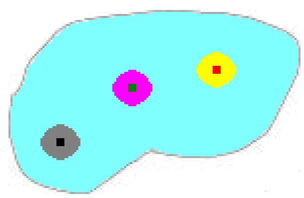
Open subsets in Euclidean spaces

It is not particularly difficult to discuss differentiation at endpoints of an interval in a line because there is only one direction in which one can differentiate. However, in two or more dimensions there are infinitely many possible directions to consider, and this complicates any possible discussion of differentiation at boundary points, even in simple, familiar cases. Therefore in the theory of functions of several variables it is customary to focus on examples where there are no boundary points on the set U for which the function is defined. Sets with this property are called open sets. The formal definition appears in Unit I of the course notes. As noted in **http://en.wikipedia.org/wiki/Open_set**,

> Intuitively speaking, a set U is open if every point $\mathbf{x}$ in U can be moved by a small amount in any direction and still be in the set U.

An example of an open subset (or region) in the plane is illustrated below. The region is the figure shaded in various colors, and three points in it are marked in black, green and red. For each point, there is a small open disk neighborhood with a positive radius (colored in **gray** for the **black** point, **pink** for the **green** point, and **yellow** for the **red** point) which is totally contained in the given region; the boundary points in light gray are **not** part of the open set. Note that the radii of the open disk neighborhoods usually vary from one point to another.



Recognizing open subsets. Given an arbitrary subset of the coordinate plane or 3 – space, one obvious question is to determine whether the subset is open. In many cases it is difficult or tedious to do this using the definition directly, but fortunately there are some simple criteria that can be used to verify that a subset is open:

1. If g is a real valued continuous function defined for all real numbers and a is a real number, then the set W_a^{+} of points where $g(\mathbf{x}) > a$ is an open subset.
2. If the set U is the intersection of two open subsets V and W, then U is also an open subset.
3. If the set U is the union of two open subsets V and W, then U is also an open subset.
4. If the set U is obtained from the open subset V by deleting finitely many points, then U is also an open subset.

Sets which are NOT open. It is also useful to recognize a few examples of subsets which are not open.

1. A nonempty finite collection of points is never an open subset.
2. A line is not an open subset.
3. In 3 – space, a plane is not an open subset.

4. A half – plane in the plane defined by a nontrivial linear inequality like $x_1 \geq 0$ is never an open subset. Similarly, a half – space in 3 – space defined by a nontrivial linear inequality like $x_1 \geq 0$ is never an open subset.
5. If the set A is the nonempty intersection of two subsets B and C of any of the types described above, then A is also not an open subset.

In each of the cases described above, the subset fails to satisfy the intuitive condition described earlier; for some points $\mathbf{y}$ in each set, there are directions $\mathbf{u}$ (given by unit vectors) such that no points $\mathbf{y} + t\mathbf{u}$ (where $t > 0$) lie in that set. For example, if the set in the first case contains exactly one point then this is true for **every** direction, and in the fourth case the statement is true for *all directions whose first coordinates are* ***negative.*** The drawings below illustrate these examples, with some of the indicated directions shown in **red**.