Solution to problem I.2.5

We begin by restating the problem: *Prove that a regular smooth curve lies on a straight line if and only if there is a point that lies on all its tangent lines.* — For the sake of definiteness, we shall assume that the curve is parametrized over a closed interval of the form $[a, b]$.

The discussion below formalizes the comments in class and fills in some gaps.

The "only if" direction is fairly straightforward. Suppose that we are given a curve $\gamma(t)$ which lies on the line L of all points of the form $\mathbf{a} + u\mathbf{b}$, where $\mathbf{b}$ is some unit vector and u is a scalar (we can always express the points on a line this way). We can then write $\gamma(t) = \mathbf{a} + u(t)\mathbf{b}$ for some function $u(t)$. The function $u(t)$ has a continuous derivative because it is given by the formula

$$u(t) \quad = \quad (\gamma(t) - \mathbf{a}) \cdot \mathbf{b}$$

and therefore we can use the Chain Rule to write $\gamma'(t) = u'(t)\mathbf{b}$. Since we have a regular curve it follows that $u'(t)$ is never zero. Therefore it follows that the tangent line to $\gamma(t)$ contains the point $\mathbf{a} + u(t)\mathbf{b}$ on L and has the same direction as L, and hence this tangent line must be L. Thus every point of the curve lies on the tangent lines for all lines on the curve.

The "if" direction is more complicated. First of all, we claim it is enough to prove the result when the point in question is the origin $\mathbf{0}$; in the general case when the point is some vector $\mathbf{q}$, all we have to do is replace the given curve $\gamma(t)$ by $\gamma(t) - \mathbf{q}$, for if the latter curve lies on a line then so does the original curve.

There are now two cases:

(1) The point $\mathbf{0}$ lies on the curve. — In this case, we shall choose the parametrization so that $\gamma(0) = \mathbf{0}$

(2) The point $\mathbf{0}$ does not lie on the curve. — In this case, we shall choose a typical point $\mathbf{p}$ on the curve and a parametrization so that $\gamma(0) = \mathbf{p}$.

In both cases we make the following CLAIM: *There is some $h > 0$ such that $\gamma(t) \neq \mathbf{0}$ for all t such that $0 < |t| < h$.*

In the second case, this is simply a consequence of continuity, for the latter implies that there is some h such that $|t| < h$ implies that $|\gamma(t)| > \frac{1}{2}\mathbf{p}$. Suppose now that we are in the first case where $\gamma(0) = \mathbf{0}$. Since we have a regular curve, it follows that $\gamma'(0) \neq \mathbf{0}$, and hence at least one of the coordinate derivatives $x'(0)$, $y'(0)$, $z'(0)$ is nonzero. By standard results from single variable calculus, if f is a continuously differentiable function such that $f(0) = 0$ and $f'(0) \neq 0$, then there is some $h > 0$ such that $f(t) \neq 0$ for all t such that $0 < |t| < h$.

For each values of t as above, we know that the tangent line at $\gamma(t)$ contains $\mathbf{0}$, and since $\gamma(t) \neq \mathbf{0}$ it follows that the line joining these points is the tangent line to γ at parameter value t. Therefore the tangent line at $\gamma(t)$ must be the 1-dimensional vector subspace spanned by $\gamma(t)$. On the other hand, since $\gamma'(t)$ is a direction vector for the tangent line, it follows that the vectors $\gamma(t)$ and $\gamma'(t)$ must be scalar multiples of each other.

Choose $f(t)$ such that $\gamma(t) = f(t) \cdot \gamma'(t)$ for all relevant values of t. The function $f(t)$ is continuous because

$$f(t) \quad = \quad \frac{\gamma(t) \cdot \gamma'(t)}{\gamma'(t) \cdot \gamma'(t)}$$

and of course $f(t)$ is nonzero unless $\gamma(0) = \mathbf{0}$. If $g = 1/f$ when $f(t) \neq 0$, then it follows that $\gamma'(t) = g(t) \cdot \gamma(t)$.

If we now choose $G(t)$ such $G'(t) = g(t)$, then the equation at the end of the previous paragraph yields the differential equation

$$\frac{d}{dt}\left(e^{-G(t)} \cdot \gamma(t)\right) \quad = \quad \mathbf{0} \ .$$

This equation is valid for $0 < |t| < h$, or equivalently when either $-h < t < 0$ or $0 < t < h$. It follows that on each of these intervals the function $e^{-G(t)} \cdot \gamma(t)$ is a constant vector, say $\mathbf{v}_{\pm}$. Therefore on the given intervals we have $\gamma(t) \ = \ e^{G(t)} \cdot \mathbf{v}_{\pm}$.

Therefore we see that all points of the form $\gamma(t)$ for $0 < t < h$ lie on a line, and likewise for all points of the form $\gamma(t)$ for $-h < t < 0$. If $\mathbf{T}(t)$ be the unit tangent vector for $\gamma(t)$, this means that $\mathbf{T}(t) \ = \ |\mathbf{v}_+|^{-1}\mathbf{v}_+$ for $0 < t < h$ and $\mathbf{T}(t) \ = \ |\mathbf{v}_-|^{-1}\mathbf{v}_-$ for $-h < t < 0$. Since $\mathbf{T}(t)$ is a continuous function defined on the entire interval $(-h, h)$, it follows that $\mathbf{T}(t)$ must be a constant function such that

$$|\mathbf{v}_+|^{-1}\mathbf{v}_+ \quad = \quad |\gamma'(0)|^{-1}\gamma'(0) \quad = \quad |\mathbf{v}_-|^{-1}\mathbf{v}_-$$

and hence all points of $\gamma(t)$ for $|t| < h$ must lie on the line joining $\gamma(0)$ and $\gamma'(0)$.

The preceding argument yields a **local** form of the stated result. We need to prove that the same conclusion holds **globally** for all t. This can be done in several different ways, and our approach will involve a closer analysis of the preceding argument.

Suppose first that $\mathbf{0}$ is not a point on the curve. Then there is no need to worry about finding some $h > 0$, for the function $f(t)$ can be defined over the entire interval on which the curve γ is defined. If there is only one value t_0 of t such that $\gamma(t) = 0$, then once again the function $f(t)$ can be defined for all $t \neq t_0$.

More generally, our arguments show that the set of all t such that $\gamma(t) = 0$ consists of *isolated* points, for if $\gamma(t) = \mathbf{0}$ then there is some $h_t > 0$ such that $\gamma(u)$ is nonzero for $0 < |u - t| < h_t$ (in other words, all points on the interval $(t - h_t, t + h_t)$ except t itself). A fundamental property of the real numbers (one variant of the Bolzano-Weierstrass Theorem) states that an isolated set of points on a closed interval must be finite, and therefore we know that the set of all t such that $\gamma(t) = \mathbf{0}$ must be finite. Write these points in order as $t_0 < t_1 < ... < t_m$.

By the previous discussion, we know that the image of γ restricted to each interval $[a, t_1)$, (t_0, t_2), and so on up to $(t_{m-1}, b]$ is contained in a line. We claim the common line is the same for each interval. This is essentially an argument by mathematical induction. Less formally, one proceeds as follows: If L_1 is the line containing the restriction of γ to $[a, t_1)$ and L_2 is the line containing the restriction of γ to (t_0, t_2), then $L_1 = L_2$ because both contain the restriction of γ to (t_0, t_1) and this image is not a single point because γ' is never zero. Thus the restriction of γ to $[a, t_2)$ is contained in a single line. One can now use the same sort of argument to show that the restriction of γ to $[a, t_3)$ is contained in a single line and so on, ending with the conclusion that the image of the entire curve — which is by definition its own restriction to the entire interval $[a, b]$ — is contained in a single line.