Supplement to I.5 — Note on the Frenet-Serret formulas

This note provides some additional information about the theorem on the existence of curves with prescribed curvature and torsion and an initial Frenet trihedron. As indicated in the notes, the first step is to find a solution to the system of 9 linear differential equations corresponding to the Frenet-Serret formulas:

$$\begin{aligned} \mathbf{T}' &= && \kappa\mathbf{N} && \\ \mathbf{N}' &= -\kappa\mathbf{T} && && -\tau\mathbf{B} \\ \mathbf{B}' &= && \tau\mathbf{N} && \end{aligned}$$

Once we have a solution with initial Frenet trihedron $\mathbf{T}_0$, $\mathbf{N}_0$, $\mathbf{B}_0$, we would like to know that the three functions $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$, are orthonormal everywhere and not just when $s = s_0$. This is done on page 310 of do Carmo in a rather terse manner, and the purpose here is to give another argument which uses matrices and hopefully sheds more light on why the solutions form an orthonormal set.

Let $A(s)$ be the 3×3 matrix whose columns are the vector valued functions $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$. Then the Frenet-Serret formulas can be rewritten in the matrix form

$$A'(s) \quad = \quad C(s)A(s)$$

where differentiation is done coordinatewise and $C(s)$ is the following matrix:

$$\begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

Note that the matrix $C(s)$ is *skew symmetric*; in other words, it is equal to the negative of its own transpose. We shall denote the transpose of a matrix P by P^* here to avoid typesetting problems.

Now define the "Gram matrix" $G(s) = A(s)^* \cdot A(s)$. We can differentiate $G(s)$ in the usual fashion, and in fact this differentiation satisfies the following two simple conditions:

(i) The derivative of a transpose is the transpose of the derivative.

(ii) There is a Leibniz rule of the form $(PQ)' = P'Q + PQ'$. One must be careful to keep the proper ordering of the factors since matrix multiplication is not commutative.

We may now use both rules to compute this derivative as follows:

$$G'(s) \quad = \quad A'(s)^*A \; + \; A(s)^*A'(s) \quad = \quad A(s)^*C(s)^*A \; + \; A(s)^*C(s)A(s)$$

Since $C(s)$ is skew-symmetric, we have $C^* = -C$ and therefore the right hand side reduces to zero.

It follows that $G'(s)$ is zero and hence $G(s)$ is constant. Since $G(s_0)$ is the identity, it follows that $G(s) = I$ for all s. By the definition of $G(s)$ in terms of inner products, this means that the columns of the orignal matrix $A(s)$ are orthonormal, and therefore it follows that the solutions to the Frenet-Serret differential equations yield an orthonormal set of vectors, which is what we wanted to show.■