Riemannian metrics and hyperbolic geometry: I

In Section III.4 of the lecture notes we mentioned that the non-Euclidean plane discovered independently by L. Bólyai, N. I. Lobachevsky and C. F. Gauss in the 19^{th} century has a natural interpretation in terms of riemannian metrics. Specifically, one takes the underlying space U to be the open unit disk about the origin in $\mathbf{R}^2$, and the riemannian metric given to the so-called **Poincaré disk metric:**

$$\frac{dx\,dx \;+\; dy\,dy}{(1-x^2-y^2)^2}$$

In the notes we described the curves of shortest length (with respect to this metric) that join pairs of points in U. The purpose here is to explain the connection between this object and non-Euclidean geometry in terms of congruence.

General considerations

The classical geometric notion of congruence has two basic properties:

(1) Given points $\mathbf{x}$ and $\mathbf{y}$ in U, there is a rigid motion (which must be an isometry) taking $\mathbf{x}$ to $\mathbf{y}$.

(2) Given a point $\mathbf{p}$ in U and two unit tangent vectors $\mathbf{x}$ and $\mathbf{v}$ at $\mathbf{p}$ there is an isometry T which sends $\mathbf{p}$ to itself and sends the curve $\gamma(t) = \mathbf{p} + t\mathbf{x}$ to the curve $T\circ\gamma$ such that $(T\circ\gamma)'(0) = \mathbf{y}$.

The notions of isometry and unit vector should be interpreted within the framework of riemannian metrics, and in this connection we use the following characterization of (smooth) riemannian isometries:

PROPOSITION. *Suppose that we are given a riemannian metric g over a connected domain in $\mathbf{R}^2$; by construction, if $u \in U$ then this yields an inner product g_u on $\mathbf{R}^2$ such that the Gram matrix coefficients*

$$g_{i,j}(u) \quad = \quad g_u(\mathbf{e}_i,\, \mathbf{e}_j)$$

are smooth functions of u (continuous first partial derivatives at least). Suppose that $f : U \to U$ is a smooth $1-1$ onto map with a smooth inverse such that for all $(u, \mathbf{x}, \mathbf{y}) \in \mathbf{R}^n$ we have the following identity:

$$g_u(\mathbf{x}, \mathbf{y}) \quad = \quad g_{f(u)}\big(\, Df(u)\mathbf{x},\, Df(u)\mathbf{y}\,\big)$$

Then for each smooth curve γ from the closed interval $[a, b]$ to U, the lengths of γ and $f\circ\gamma$ are equal. In particular, if $\mathbf{p}$ and $\mathbf{q}$ are in U and γ is a smooth curve of shortest length joining $\mathbf{p}$ and $\mathbf{q}$, then $f\circ\gamma$ is a smooth curve of shortest length joining $f(\mathbf{p})$ and $f(\mathbf{q})$.

The equality of length follows directly from the isometry identity, and the statement about curves of shortest length follows because f preserves lengths of curves.∎

Definition. A map f satisfying the condition in the proposition will be called a *riemannian isometry.* If we are working with riemannian isometries, then the second condition involving congruence can be reformulated as follows:

(2′) Given a point $\mathbf{p}$ in U and two unit tangent vectors $\mathbf{x}$ and $\mathbf{y}$ at $\mathbf{p}$ there is a riemannian isometry f which sends $\mathbf{p}$ to itself and satisfies $Df(\mathbf{p})\mathbf{x} = \mathbf{y}$.

It will also be helpful to have the following general facts about riemannian isometries.

THEOREM. *Suppose that U is as above.*

(i) If f and g are riemannian isometries of U, then so is their composite $g \circ f$, and the inverses of f and g are also riemannian isometries.

(ii) Suppose that $\mathbf{p} \in U$ and f is a riemannian isometry of U. Suppose that condition $\mathbf{(2')}$ above is satisfied at $\mathbf{p}$, let $\mathbf{q}$ be some other point of U, and suppose there is a riemannian isometry h of U such that $h(\mathbf{p}) = \mathbf{q}$. Then condition $\mathbf{(2')}$ above is satisfied at $\mathbf{q}$.

Sketch of proofs. The first part is a routine computation and is left to the reader as an exercise.

To prove the second part, let $\mathbf{u}$ and $\mathbf{v}$ be unit tangent vectors at $\mathbf{q}$. Then $\mathbf{x} = Df^{-1}(\mathbf{q})\mathbf{u}$ and $\mathbf{y} = Df^{-1}(\mathbf{q})\mathbf{v}$ are unit tangent vectors at $\mathbf{p}$, so by the hypothesis there is some riemannian isometry h of U which maps $\mathbf{p}$ to itself and satisfies $Dh(\mathbf{p})\mathbf{x} = \mathbf{y}$. If we take $g = f \circ h \circ f^{-1}$, then direct computation shows that g will take $\mathbf{q}$ to itself and satisfy $Dg(\mathbf{q})\mathbf{u} = \mathbf{v}$.■

Application to the Poincaré metric

To complete the linkage of the Poincaré metric with non-Euclidean geometry, we need to check that it satisfies property $\mathbf{(1)}$ above and that it also satisfies property $\mathbf{(2')}$ at some point of the open unit disk. By the previous theorem, these will imply that the metric also satisfies property $\mathbf{(2')}$ at **every** point of the open unit disk.

Verification of property $\mathbf{(2')}$ at the origin $\mathbf{0}$. This turns out to be remarkably simple. Given a 2×2 orthogonal matrix A, we know that it maps the open unit disk to itself, so let f_A be this 1–1 onto mapping. It clearly has an inverse whose coordinate functions have continuous partials. Furthermore, direct calculation shows that f_A is a riemannian isometry with respect to the Poincaré metric.

Now unit vectors for the Poincaré metric at the origin are just unit vectors with respect to the standard inner product on $\mathbf{R}^2$. Given two such unit vectors, there is an orthogonal matrix taking one to the other, and since $Df_A(\mathbf{0}) = A$ it follows that property $\mathbf{(2')}$ is satisfied at $\mathbf{0}$.

Verification of property $\mathbf{(1)}$. This is more difficult, so we shall first chip away at it with a sequence of reductions.

(a) It suffices to verify the property when one of the points is the origin. Suppose we know the property holds in this case, and let $\mathbf{p}$ and $\mathbf{q}$ be arbitrary points in the open disk. Since we are assuming the condition in the reduction, there are riemannian isometries f and g such that $f(\mathbf{0}) = \mathbf{p}$ and $g(\mathbf{0}) = \mathbf{q}$; the composite $h = g \circ f^{-1}$ then maps $\mathbf{p}$ to $\mathbf{q}$.

(b) It suffices to verify the property when one of the points is the origin and the other is on the positive x-axis. Suppose we know the property holds in such cases, and suppose that we have a nonzero point $\mathbf{q}$ on the unit disk. We may then write $\mathbf{q} = t\mathbf{v}$, where $\mathbf{v}$ is a unit vector and $0 < t < 1$. Since we are assuming the condition in the reduction, we have a riemannian isometry h which maps $\mathbf{0}$ to $t\mathbf{e}_1$. However, we also have an orthogonal matrix A which sends $\mathbf{e}_1$ to $\mathbf{v}$, and if f_A is the associated riemannian isometry then it will follow that $f_A \circ h$ will send $\mathbf{0}$ to $\mathbf{v}$.

(c) Finding a riemannian isometry which sends the origin to $t\mathbf{e}_1$, where t is an arbitrary number strictly between 0 and 1. This is by far the least obvious step in the whole process, and it is best

done using complex numbers. Consider the following so-called Möbius function, which is a quotient of two linear functions defined for all complex numbers x:

$$f(z) \quad = \quad \frac{az+b}{bz+a} \qquad \text{where} \qquad a \quad = \quad \frac{1}{\sqrt{1-t^2}} \qquad \text{and} \qquad b \quad = \quad \frac{t}{\sqrt{1-t^2}}$$

This complex valued function is defined for all values of z except $-1/t$, and since $0 < t < 1$ it is defined on the open unit disk U. By construction we have $f(0) = t$ and $a^2 - b^2 = 1$.

It is probably not obvious that f sends the open unit disk U into itself. The reasons for this involve depend upon the fact that $a^2 - b^2 = 1$, and they are described in the section of the following online document titled *Disk model actions*:

`http://en.wikipedia.org/wiki/Hyperbolic_motion`

Verifying that f is a riemannian isometry requires a direct calculation of Df, which is elementary but messy. We shall omit the details.■

The preceding discussion shows that the Poincaré metric has an extensive collection of riemannian isometries, and from the viewpoint of differential geometry this is one of the key ties to non-Euclidean geometry.

The online sites listed below contain more informaion on the interpretation of non-Euclidean geometry using differential geometry.

`http://en.wikipedia.org/wiki/Hyperbolic_geometry`

`http://en.wikipedia.org/wiki/Upper_half_plane`

`http://mathworld.wolfram.com/PoincareHyperbolicDisk.html`