Mathematics 138A, Winter 2006, Examination 1

This will be due at the beginning of class on
Wednesday, February 15, 2006.

You must show the work behind or reasons for your answers. The point values are indicated in parentheses.

1. (*20 points*) Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be orthonormal vectors in $\mathbf{R}^3$ such that $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ (cross product). Compute $\mathbf{v} \times \mathbf{w}$ and $\mathbf{w} \times \mathbf{u}$.

2. (*30 points*) (*a*) If an object is attached to the edge of a circular wheel and the wheel is rolled along a straight line on a flat surface at a uniform speed, then the curve traced out by the object is a **cycloid** (there is an illustration in the files `exam1figures.*`). If the circle has radius $a > 0$ and its center starts at the point with coordinates $(0, a)$, then the object starts at $(0, 0)$ and its parametric equations are given by the classical formula $\mathbf{x}(t) = a \cdot (t - \sin t, 1 - \cos t)$. Find the length of the cycloid over the parameter values $0 \leq t \leq 2\pi$.

(*b*) In the classical geocentric theory of planetary motion which appears in the *Almagest* of Claudius Ptolemy (*c.* 85–165), there is an assumption that planets travel in curves given by *epicycles*. The simplest examples of these involve circular motion where the center of the circle is moving in a circular path around a second circle (this is similar to the motion of the moon around the earth, which is given by an ellipse while the earth itself is moving around the sun by a larger ellipse; an illustration appears in `exam1figures.*`; in the full theory one also allowed the second circle to move around a third circle, and so on). A typical example is given by the following formula, in which the first circle has radius $\frac{1}{4}$, the second one is the unit circle about the origin, and the body rotates four times around the small circle as the large circle makes one revolution around its center:

$$\mathbf{x}(t) \quad = \quad (\cos t, \sin t) \; + \; \tfrac{1}{4}\,(\cos 4t, \sin 4t)$$

Find the length of this curve over the parameter values $0 \leq t \leq 2\pi$.

Note. For both parts of these exercises the standard formulas for $|\sin \frac{1}{2}\theta|$ and $|\cos \frac{1}{2}\theta|$ may be useful.

3. (*20 points*) Find the curvature and torsion for the general helix defined by the equation $\mathbf{x}(t) = (a\cos t, a\sin t, bt)$, where $a > 0$ and $b \neq 0$.

4. (*30 points*) Let f be a real valued function on $\mathbf{R}^3$ with continuous second partial derivatives, and let $\mathbf{G}$ be a 3-dimensional vector valued function on $\mathbf{R}^3$ (often called a *vector field*) with continuous second partial derivatives. Let ω be the differential 1-form corresponding to $\mathbf{G}$.

(*a*) Express the differential 2-form θ corresponding to the vector field $\nabla f \times \mathbf{G}$ in terms of f and ω.

(*b*) Given a differential 1-form $\omega = P\,dx + Q\,dy + R\,dz$ and a differential 2-form

$$\theta = L\,dy \wedge dz + M\,dz \wedge dx + R\,dx \wedge dy$$

the wedge product $\omega \wedge \theta$ is a differential 3-form which may be written as $h\,dx \wedge dy \wedge dz$ for some real valued function h. Express h in terms of the functions L, M, N, P, Q, R.

(*c*) Using differential forms, derive a formula for the divergence of $\nabla f \times \mathbf{G}$ involving some (but not necessarily all) of ∇f, $\nabla^2 f$, $\operatorname{div} \mathbf{G} = \nabla \cdot \mathbf{G}$, and $\operatorname{curl} \mathbf{G} = \nabla \times \mathbf{G}$.

References. In addition to Section II.1 of the notes, the basic facts about differential forms also appear on pages 30–33 of O'Neill.