Mathematics 138A, Winter 2006, Examination 2

This will be due at the beginning of class on
Wednesday, March 8, 2006.

You must show the work behind or reasons for your answers. The point values are indicated in parentheses.

1. (*20 points*) (*a*) An invertible $n \times n$ matrix A is said to be *conformal* if it preserves angles; *i.e.*, if $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbf{R}^n$ then

$$\cos \angle(\mathbf{x}, \mathbf{y}) \quad = \quad \cos \angle(A\mathbf{x}, A\mathbf{y})$$

where the cosine may defined by the usual inner product formula.

(*a*) Suppose that $A = c\,B$ where B is orthogonal and $c > 0$. Show that A is conformal.

(*b*) Suppose that A is conformal. Prove that the columns of A are perpendicular. [*Hint:* They define the vectors $A\mathbf{e}_i$ where the $\mathbf{e}_i$ are the standard unit vectors in $\mathbf{R}^n$.]

(*c*) Suppose that L_i is the (positive) length of $A\mathbf{e}_i$. Compute L_i/L_1. [*Hint:* look at the angle between $\mathbf{e}_1 + \mathbf{e}_i$ and $\mathbf{e}_1$ and the angle between the images of these vectors under A.]

(*d*) Why do the preceding two parts of the problem imply that if A is conformal then $A = cB$ where B is orthogonal and $c > 0$?

(*e*) Let f be the map from $\mathbf{R}^2$ to itself defined by $f(u,v) = (u^2 - v^2, 2uv)$. Prove that $Df(u,v)$ is conformal for all $(u,v) \neq (0,0)$. [*Hint:* Set $u = r\cos\theta$ and $v = r\sin\theta$ in $Df(u,v)$.]

2. (*30 points*) Let $f(u,v,w) = uvw - 1$.

(*a*) Show that the set of points where $f(u,v,w) = 0$ defines a surface V_f. [*Hint:* If a product of a finite sequence of numbers is nonzero, why must each of the terms in the sequence be nonzero?]

(*b*) Let (a,b,c) be positive constants. Find all points $(x,y,z) \in V_f$ such that the tangent plane to V_f at (x,y,z) is parallel to the plane $au + bv + cw = 0$.

3. (*20 points*) (*a*) Let $\mathbf{a}(t)$ be a regular smooth plane curve (so its third coordinate vanishes). Then the cone on $\mathbf{a}$ with vertex equal to the third unit vector $\mathbf{e}_3 = (0,0,1)$ is given by the parametrization $(1-v)\mathbf{a}(u) + v\mathbf{e}_3$ or equivalently in the ruled surface form $\mathbf{a}(u) + v(\mathbf{e}_3 - \mathbf{a}(u))$. Show that this is a regular parametrization for $v < 1$.

(*b*) In the preceding example, show that the tangent planes are the same for all points on the rulings $\mathbf{a}(u_0) + v(\mathbf{e}_e - a(u_0))$, where u_0 is fixed and v is allowed to vary from $-\infty$ to 1. [*Hint:* Why does the tangent plane at a point contain all points on the ruling through that point? Recall that there is a unique plane containing a given point and perpendicular to a given direction.]

4. (*30 points*) (*a*) Suppose that f is an isometry of $\mathbf{R}^3$ and $\mathbf{x}$ is a regular parametrization of a surface which is defined at a point $\mathbf{p}$. Explain why the tangent plane to the surface for $f \circ \mathbf{x}$ at $f(\mathbf{p})$ is the image under f of the tangent plane to the surface for $\mathbf{x}$ at $\mathbf{p}$. [*Hint:* Write $f(v) = A\mathbf{v} + \mathbf{b}$.]

(*b*) Let h be the dilation map from $\mathbf{R}^3$ to itself given by $h(\mathbf{x}) = c\mathbf{x}$, where $c > 0$ and $c \neq 1$, and let $\mathbf{x}$ and $\mathbf{p}$ be as in the previous part of the problem. What is the relationship between the tangent planes to the surface for $f \circ \mathbf{x}$ at $f(\mathbf{p})$ is the image under f of the tangent plane to the surface for $\mathbf{x}$ at $\mathbf{p}$?

(*c*) Given planes P_1 and P_2 in $\mathbf{R}^3$, there are three possibilities. We may have $P = Q$, we may have that $P \cap Q$ is a line, or else P and Q might be parallel. Which, if any, of these cannot occur for the tangent planes in the second part of the problem? Give reasons for your answer.