Mathematics 138A, Winter 2006, Examination 3, Part 1

This will be due at the start of the second part of the examination on Thursday, March 23, 2006, at 9:00 A.M.

The total number of possible points for this part of the examination is 75, and the number for the second part will be 125.

You must show the work behind or reasons for your answers. The point values are indicated in parentheses.

1. (*40 points*) As in the second examination, let $\mathbf{a}(t)$ be a regular smooth plane curve (so its third coordinate vanishes). Then the cone on $\mathbf{a}$ with vertex equal to the third unit vector $\mathbf{e}_3 = (0,0,1)$ is given by the parametrization $(1-v)\mathbf{a}(u) + v\mathbf{e}_3$ or equivalently in the ruled surface form $\mathbf{a}(u) + v(\mathbf{e}_3 - \mathbf{a}(u))$. By one of the problems in the second examination, this is a regular parametrization for $v < 1$.

(*a*) Compute the local First and Second Fundamental Forms in terms of the given parametrization. The answer should be expressed in terms of the variables u and v, the vector valued functions $\mathbf{a}(u)$ and $\mathbf{b}(u)$, and the first and second derivatives of these functions.

(*b*) Compute the Gaussian curvature for the surface described above in terms of the same quantities as in (*a*).

(*c*) Compute the mean curvature if $\mathbf{a}$ is the circle $(c + r\cos t, r\sin t)$, where $r > 0$ and c is a real number, and also for the piece of the parabola $y = x^2 - 1$ for $|x| \leq 1$.

(*d*) Set up the integrals for computing the areas of the portions of the surfaces in (*c*) lying between the planes $z = 1$ and $z = 0$. You do not need to evaluate the integrals.

2. (*35 points*) Let A be a symmetric 2×2 matrix. An *asymptotic vector* for A is a nonzero vector $\mathbf{v}$ such that $\langle A\mathbf{v}, \mathbf{v}\rangle = 0$.

(*a*) Suppose that A is an invertible matrix such that $\det A > 0$. Explain why A cannot have any asymptotic vectors. [*Hint:* Use Rayleigh's principle. What can one say about the eigenvalues of A in this case using the positivity of $\det A$?]

(*b*) Suppose now that A is invertible and $\det A < 0$. Prove that A has two linearly independent asymptotic vectors such that every asymptotic vector is a multiple of one of these vectors. [*Hint:* Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be orthonormal eigenvectors for A with corresponding eigenvalues $\lambda_1 \geq \lambda_2$. What does the determinant condition imply about the signs of these vectors? Given a nonzero vector $\mathbf{v} = x\mathbf{u}_1 + y\mathbf{u}_2$, find a necessary and sufficient condition on x and y for $\mathbf{v}$ to be an asymptotic vector in terms of λ_1 and λ_2. You should get conditions of the form $y = \pm\, c\, x$ for some nonzero constant c. What is it?]

(*c*) Suppose that A as in the preceding part of the problem and $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent asymptotic vectors for A. Express the absolute value of the cosine of the angle between these vectors in terms of the eigenvalues. [*Note:* Since $\mathbf{v}_1$ and $-\mathbf{v}_2$ are also linearly independent asymptotic vectors, only the absolute value of the cosine is independent of the choices for $\mathbf{v}_1$ and $\mathbf{v}_2$.]

(*d*) Continuing with the setting of the previous two parts of the problem, give a necessary and sufficient condition on A for asymptotic vectors to be perpendicular to each other.

This completes the description of the problems for the first part of the examination. An explanation of the relevance of this to differential geometry is given on the next page.

Motivation for Problem 2. The relevance to differential geometry arises from the notion of *asymptotic curves* in a surface with negative Gaussian curvature. These curves, which play an important role in the study of negatively curved surfaces, are regular smooth curves γ in an oriented surface such that for each parameter value t we have

$$0 \quad = \quad \langle \mathbf{S}_{\gamma(t)}\, \gamma'(t),\, \gamma'(t) \rangle$$

where (as usual) $\mathbf{S}$ denotes the shape operator for the oriented surface. Further information on such curves (and many other important types of curves in a surface) is given in O'NEILL.