Solutions to Examination 1

1. These are applications of the "BAC — CAB" Rule. In particular, we have

$$\mathbf{v} \times \mathbf{w} \;=\; \mathbf{v} \times \left(\mathbf{u} \times \mathbf{v}\right) \;=\; (\mathbf{v} \cdot \mathbf{v})\,\mathbf{u} \;-\; (\mathbf{u} \cdot \mathbf{v})\,\mathbf{v} \;=\; \mathbf{u}\ .$$

One can prove the second result in a similar fashion by considering

$$\left(\mathbf{u} \times \mathbf{v}\right) \times \mathbf{v} \;=\; -\mathbf{v} \times \left(\mathbf{u} \times \mathbf{v}\right)$$

or else by simply applying the first case to the vectors $\mathbf{u}' = \mathbf{v}$, $\mathbf{v}' = \mathbf{w}$ and $\mathbf{w}' = \mathbf{u}$. We then have $\mathbf{u}' \times \mathbf{v}' = \mathbf{w}'$, so the first part of the problem then implies that $\mathbf{v}' \times \mathbf{w}' = \mathbf{u}'$. If we translate this back into a statement about the original vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, then we conclude that $\mathbf{w} \times \mathbf{u} = \mathbf{v}$.∎

2. (*a*) The arc length is given by $\int_0^{2\pi} a|\mathbf{x}'(t)|\,dt$. If we expand $|\mathbf{x}'|$ using the parametric equations, we see that it is equal to $a\sqrt{2 - 2\cos t}$, and by the half angle formulas in trigonometry this is equal to $2a|\sin \frac{1}{2}t|$; note the need to take absolute values because the integrand is nonnegative and the sine function has both negative and positive values. On the other hand, since $\sin \frac{1}{2}t$ is nonnegative for t between 0 and 2π, we can suppress the signs in the integral formula and conclude that thus the arc length of the given piece of the cycloid curve is equal to

$$\int_0^{2\pi} 2a \sin \tfrac{1}{2}t\; dt \;=\; 4a \cdot \int_0^{\pi} \sin u\; du \;=\; 8a\ .$$

Incidentally, this formula has been attributed to the English architect Christopher Wren (1632–1723).∎

(*b*) Once again it is necessary to expand $|\mathbf{x}'|$ using the parametric equations for the curve, and this time we find that the integrand of the arc length formula is equal to

$$\sqrt{2 + 2\cos t \cos 4t + 2 \sin t \sin 4t} \;=\; \sqrt{2 + 2\,\cos(4t - t)} \;=\; \sqrt{2 + 2\,\cos 3t}$$

which by the half angle formulas in trigonometry is equal to $2\left|\cos \frac{3}{2}t\right|$; if we had chosen the radius of the smaller circle to be something other than $\frac{1}{4}$, then the integrand would have almost certainly been something that could not be handled by the usual methods for finding antiderivatives, but fortunately that is not an issue here. This time we have to be careful about the sign of the integrand, and if we use the fact that $\cos \frac{3}{2}t$ is nonnegative when $0 \leq t \leq \frac{1}{3}\pi$ or $\pi \leq t \leq \frac{5}{3}\pi$, while it is negative if $\frac{1}{3}\pi \leq t \leq \pi$ or $\frac{5}{3}\pi \leq t \leq 2\pi$, then we see that

$$\int_0^{2\pi} 2\left|\cos \tfrac{3}{2}t\right|\; dt \;=$$

$$\int_0^{\pi/3} 2\cos \tfrac{3}{2}t\, dt \;-\; \int_{\pi/3}^{\pi} 2\cos \tfrac{3}{2}t\, dt \;+\; \int_{\pi}^{5\pi/3} 2\cos \tfrac{3}{2}t\, dt \;-\; \int_{5\pi/3}^{2\pi} 2\cos \tfrac{3}{2}t\, dt\ .$$

If we evaluate the integrals separately and combine them as indicated, we find that the length of the curve is equal to 8 for this example (just as it was for the first part of the problem!).∎

3. We shall follow the procedure used in the document `helix.pdf`. The parametric equations imply that $\mathbf{x}'(t) = (-a\sin t, a\cos t, b)$, so that the length is given by $|\mathbf{x}'(t)| = \sqrt{a^2 + b^2} = s'(t)$ and consequently we may take the arc length parametrization to be given by $s = \sqrt{a^2+b^2}\cdot t$ or $t = s/\sqrt{a^2+b^2}$. One then has the following formula for the unit tangent vector:

$$\mathbf{T}(t) \quad = \quad \frac{1}{\sqrt{a^2+b^2}}\cdot(\,-a\sin t, \cos t, b\,)$$

It follows that the curvature is given by

$$\kappa(t) \quad = \quad \frac{|\mathbf{T}'(t)|}{s'(t)} \quad = \quad \frac{a}{a^2+b^2}$$

and the principal unit normal is given as follows:

$$\mathbf{N}(t) \quad = \quad (-\cos t, -\sin t, 0)$$

If we compute $\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t)$ we obtain the formula

$$\mathbf{B}(t) \quad = \quad \frac{1}{\sqrt{a^2+b^2}}\cdot(\,b\sin t, -b\cos t, a\,)$$

so that we havso that we have

$$\mathbf{B}'(s) \quad = \quad \frac{1}{s'(t)}\cdot\mathbf{B}'(t) \quad =$$

$$\frac{1}{a^2+b^2}\cdot(\,b\cos t, b\sin t, 0\,) \quad = \quad \frac{-b}{a^2+b^2}\cdot\mathbf{N}(s)$$

which means that the torsion of the curve is given by

$$\tau \quad = \quad \frac{-b}{a^2+b^2}$$

at every point of the curve.■

P. S. Although the problem does not ask for this, it is natural to ask what constant values for curvature and torsion are realized by the curves described in the problem if a is allowed to vary over all positive real numbers and b is allowed to vary over all nonzero real numbers. It turns out that one can realize all values κ and τ such that $\kappa > 0$ and $\tau \neq 0$. — By the preceding result, this amounts to saying that for such values of κ and τ one can find $a > 0$ and $b \neq 0$ such that

$$\kappa \quad = \quad \frac{a}{a^2+b^2} \qquad \text{and} \qquad \tau \quad = \quad \frac{-b}{a^2+b^2}\ .$$

However, if we take

$$a \quad = \quad \frac{\kappa}{\kappa^2+\tau^2} \qquad \text{and} \qquad b \quad = \quad \frac{-\tau}{\kappa^2+\tau^2}$$

then by construction we have

$$a^2+b^2 \quad = \quad \frac{1}{\kappa^2+\tau^2}$$

so that $a = \kappa(a^2+b^2)$ and $b = -\tau(a^2+b^2)$. Dividing everything in sight by $(a^2+b^2)^2$, we see that a and b solve the required equations and thus yield a curve with the prescribed curvature κ and torsion τ.■

ALTERNATE APPROACH. There is also a very quick, but less elementary, way of seeing this using complex numbers. By the formulas derived in the problem we have $\kappa + i\,\tau = (a\,+\,bi)^{-1}$. Since $(z^{-1})^{-1} = z$ for all complex numbers z, this means that if we are given κ and τ and we take a and b such that $(\kappa + i\,\tau)^{-1} = a\,+\,bi$, then it will follow that $\kappa + i\,\tau = (a\,+\,bi)^{-1}$.$\blacksquare$

4. (*a*) The vector field ∇f corresponds to the 1-form df, and the cross product of vector fields corresponds to wedge product of 1-forms, so the 2-form we want is just $df \wedge \omega$.$\blacksquare$

(*b*) Using the standard rules for manipulating differential forms, one can compute the wedge product as follows:

$$\omega \wedge \theta \quad =$$

$$\big(\,P\,dx + Q\,dy + R\,dz\,\big) \wedge \big(\,L\,dy \wedge dz + M\,dz \wedge dx + R\,dx \wedge dy\big)$$

If we expand this using distributivity and recall that a wedge expression $du \wedge dv \wedge dw$ is zero if two factors are equal, then we find that the right hand side is equal to the following expression:

$$(L \cdot P)\,dx \wedge dy \wedge dz \;\;+\;\; (M \cdot Q)\,dz \wedge dx \wedge dy \;\;+\;\; (N \cdot R)\,dx \wedge dy \wedge dz$$

We now use the relations $du \wedge dv = -\,dv \wedge du$ to conclude that

$$dz \wedge dx \wedge dy \quad = \quad -\,dx \wedge dz \wedge dy \quad = \quad dx \wedge dy \wedge dz$$

$$dy \wedge dz \wedge dx \quad = \quad -\,dy \wedge dx \wedge dz \quad = \quad dx \wedge dy \wedge dz$$

and if we combine these with the formula for $\omega \wedge \theta$ we see that the latter is equal to

$$\big(\,L \cdot P \;+\; M \cdot Q \;+\; N \cdot R\,\big)\,dx \wedge dy \wedge dz\ .$$

Although this part of the problem does not ask for it directly, we can use the preceding to conclude that if ω and θ correspond to the vector fields $\mathbf{F}$ and $\mathbf{G}$ respectively, then their wedge product is merely the 3-form $(\mathbf{F} \cdot \mathbf{G})\,dx \wedge dy \wedge dz$. *This formula is needed in the third part of the problem.*$\blacksquare$

(*c*) The rules for exterior derivatives imply that $d(\,df \wedge \omega\,) = d(df) \wedge \omega - df \wedge d\omega$ and since $d(df) = 0$ we are left with only the second term. As noted in earlier parts of the problem, the 1-form df corresponds to the vector field ∇f, and if ω corresponds to the vector field $\mathbf{G}$ then $d\omega$ corresponds to the vector field $\nabla \times \mathbf{G}$. By the second part of the problem we know that the wedge of df and $d\omega$ will correspond to the dot product of ∇f and $\nabla \times \mathbf{G}$. Therefore the negative of this dot product will correspond to $d(\,df \wedge \omega\,)$, which also corresponds to the divergence of $\nabla f \times \mathbf{G}$. If we combine all these observations, we obtain the following formula:

$$\nabla \cdot \big(\nabla f \times \mathbf{G}\big) \quad = \quad -\,(\nabla f) \cdot \big(\,\nabla \times \mathbf{G}\,\big)\ \blacksquare$$

P. S. More generally, one can use the methods of this problem to compute the divergence of $\mathbf{F} \times \mathbf{G}$ in terms of the given vector fields and their curls.