More on shape operators and the Second Fundamental Form

Since the shape operator plays such a central role in O'NEILL, it seems extremely worthwhile to include a few more identities involving it that are useful for computational purposes. We shall also state explicitly the symmetry property of the Second Fundamental Form which plays a central role in Sections IV.3 – IV.5 (and also in the supplementary topics in Unit V).

Sign conventions

In O'NEILL the shape operator is given by the **negative** of the map DN discussed on page 81 of the lecture notes. However, the sign in the definition of the Second Fundamental Form is correctly given in the definition on the same page.

Formal definition of the shape operator

Since the shape operator was only defined informally in the notes, we shall begin with a more systematic approach. Since we are interested in local formulas here, we shall assume that our surface Σ is the image of some smooth parametrization $\mathbf{X}$ which is defined on a connected domain and is 1–1. Suppose that $\mathbf{N}^{\Sigma}$ is an orientation for Σ. If we take the standard orientation associated to the parametrization given by $\mathbf{N}^{\text{local}}$ which is the unit vector pointing in the same direction as

$$\frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v}$$

Then for each point $\mathbf{p} = \mathbf{X}(u,v)$ on Σ we have

$$\mathbf{N}^{\Sigma}\big(\mathbf{X}(u,v)\big) \quad = \quad \varepsilon(\mathbf{p}) \cdot \mathbf{N}^{\text{local}}(u,v)$$

where $\varepsilon = \pm 1$ and is continuous. As noted before, it follows that ε is locally constant, so at least if we cut down the domain of $\mathbf{X}$ to a small open disk containing some point (u_0, v_0) we can assume that ε is constant, and in this case for the sake of convenience we shall assume that $\varepsilon = 1$.

The definition of the shape operator involves the notion of tangent space to Σ defined at the beginning of Section III.4; by construction an element of the tangent space $T(\Sigma)$ is a pair $(\mathbf{p}, \mathbf{q})$ consisting of a point $\mathbf{p} \in \Sigma$ and a tangent vector $\mathbf{q}$ to Σ at $\mathbf{p}$ (in other words, there is a smooth curve γ in $\mathbf{R}^3$ such that $\gamma(0) = \mathbf{p}$, the image of γ is contained in Σ, and $\gamma'(0) = \mathbf{q}$). We would like to define a mapping W_0 from $T(\Sigma)$ to $\mathbf{R}^3$ with the following properties:

(1) In the setting above, W maps $(\mathbf{p}, \mathbf{q}) \in T(\Sigma)$ to

$$-\big(\mathbf{N}^{\Sigma} \circ \gamma\big)'(0)$$

where γ is given as above.

(2) The function W is linear in the second variable $\mathbf{q}$ when $\mathbf{p}$ is held constant.

(3) For each $(\mathbf{p}, \mathbf{q}) \in T(\Sigma)$ the vector $W(\mathbf{p}, \mathbf{q})$ lies in the space of tangent vectors $T_{\mathbf{p}}(\Sigma)$ to Σ at $\mathbf{p}$.

(4) The mapping W has reasonable continuity and differentiability properties.

If we have such a map W, then the SHAPE OPERATOR is formally defined by the formula

$$\mathbf{S}(\mathbf{p},\mathbf{q}) \quad = \quad (\,\mathbf{p},\, W(\mathbf{p},\mathbf{q})\,)$$

and for each $\mathbf{p}$ the associated linear map from $T_{\mathbf{p}}(\Sigma)$ to itself will be denoted by $\mathbf{S}_{\mathbf{p}}$.

The third property is shown explicitly in the notes, and we shall consider the remaining ones in order.

Independence of choice of curve. In order to show this, we shall use the parametrization. Let $\mathbf{p} = \mathbf{X}(u,v)$; the Normal Thickening Principle in Section III.2 implies that, at least locally, we can write $\gamma = \mathbf{X}\circ\alpha$ for some smooth curve α in the domain of $\mathbf{X}$ such that $\alpha(0) = (u,v)$ and $\mathbf{w} = \alpha'(0)$ satisfies $\mathbf{q} = D\mathbf{X}(u,v)[\mathbf{w}]$. It then follows from the defining formulas and the Chain Rule that

$$\left(\mathbf{N}^{\Sigma}\circ\gamma\right)'(0) \quad = \quad \left(\mathbf{N}^{\Sigma}\circ\mathbf{X}\circ\alpha\right)'(0) \quad =$$

$$\left(\mathbf{N}^{\mathrm{local}}\circ\alpha\right)'(0) \quad = \quad D\mathbf{N}^{\mathrm{local}}(u,v)\alpha'(0) \quad = \quad D\mathbf{N}^{\mathrm{local}}(u,v)\mathbf{w}\ .$$

If we have chosen some other curve γ_1 with the right properties, then we would obtain a similar curve α_1 in the domain of $\mathbf{X}$ such that $\alpha'(0) = \alpha_1'(0) = \mathbf{w}$. Therefore it follows that we would obtain the same tangent vector if we used γ_1 instead of γ, and this proves (1).■

Linearity in the second variable. This is an immediate consequence of the equations displayed above. If we have curves γ_1 and γ_2 in the surface through $\mathbf{p}$, then we have the corresponding curves α_1 and α_2 in the domain of $\mathbf{X}$ through (u,v). Consider the curve

$$\alpha_0(t) \quad = \quad \alpha_1(t)\ +\ \alpha_2(t)\ -\ (u,v)\ .$$

Then we have $\alpha_0(0) = (u,v)$ so that the curve lies in the domain of $\mathbf{X}$ for t sufficiently close to 0, and in addition we have the identity

$$\alpha_0'(0) \quad = \quad \alpha_1'(0)\ +\ \alpha_2'(0)\ .$$

If we let $\gamma_0 = \mathbf{X}\circ\alpha_0$, then the Chain rule implies

$$\gamma_0'(0) \quad = \quad \gamma_1'(0)\ +\ \gamma_2'(0)$$

and the additivity property of W follows immediately from this and the previously displayed formulas. Similarly, if we are given γ and α as before and c is a scalar, then the curve $\beta(t) = \gamma(ct)$ satisfies $\beta'(0) = c\,\gamma'(0)$ and $\beta(t) = \mathbf{X}\circ\alpha(ct)$, which implies the homogeneity of W with respect to scalar multiplication. These observations show that W is linear in $\mathbf{q}$ if $\mathbf{p}$ is held fixed.■

Continuity and smoothness properties. We shall be somewhat sketchy about these because although they may be intuitively clear, writing out all the details is lengthy and not particularly instructive; furthermore, we can often avoid using these properties directly in elementary work. Since continuity and smoothness only depend on the behavior of a function very close to an arbitrary point, we shall focus on an arbitrary point $\mathbf{p}_0 = \mathbf{X}(u_0,v_0)$ of the surface and all point sufficiently close to $\mathbf{p}_0$ so that an smooth inverse to the normal thickening map Φ (from Section III.2) can be defined. We then have a nonzero vector valued function $\mathbf{G}_0 = \Phi_1\times\Phi_2$, where Φ_j is the j^{th} partial derivative vector, and we take $\mathbf{G}$ to be the unit vector pointing in the same direction as $\mathbf{G}_0$. It follows that

$$W(\mathbf{p},\mathbf{q}) \quad = \quad -D\mathbf{N}^{\mathrm{local}}\left(\,\Phi^{-1}(\mathbf{p})\,\right)\mathbf{z}$$

where

$$\mathbf{z} \quad = \quad D\Phi^{-1}(\mathbf{p})\mathbf{q}$$

and this description yields all the continuity and smoothness properties one could hope for.■

Local formula for the shape operator

The derivation of the first property yields the following description of the shape operator in terms of the parametrization:

LOCAL FORMULA. *If* $\mathbf{p} = \mathbf{X}(u,v)$ *and* $\mathbf{q} = D\mathbf{X}(u,v)\mathbf{w}$ *in the setting above, then*

$$\mathbf{S}(\mathbf{p},\mathbf{q}) \quad = \quad \big(\,\mathbf{X}(u,v),\ -D\mathbf{N}^{\mathrm{local}}(u,v)\mathbf{w}\,\big)\ .$$

This is merely a reformulation of the previously displayed identity.■

Computing the Second Fundamental Form

Once again, the global object on Σ and the local object defined on the domain of a parametrization are often identified with each other during informal discussions, so we shall begin by describing and comparing the local and global versions.

GLOBAL VERSION. Given two points $\mathbf{p},\mathbf{a})$ and $\mathbf{p},\mathbf{b})$ in $T(\Sigma)$ representing tangent vectors to the same point, the global Second Fundamental Form is defined by

$$\mathbf{II}^{\Sigma}\big(\,(\mathbf{p},\mathbf{a}),\,(\mathbf{p},\mathbf{b})\,\big) \quad = \quad \langle\, W(\mathbf{p},\mathbf{a}),\ \mathbf{b}\,\rangle$$

where $\langle -,-\rangle$ denotes the usual inner product in $\mathbf{R}^3$ and the Second Fundamental Form is often written more compactly as as $\mathbf{II}^{\Sigma}_{\mathbf{p}}(\mathbf{a},\mathbf{b})$.

LOCAL VERSION. This is defined for all (u,v) in the domain of $\mathbf{X}$ and all vectors $\mathbf{y}$ and $\mathbf{z}$ in $\mathbf{R}^2$ by the classical formula

$$\mathbf{II}^{\mathrm{local}}_{(u,v)}(\mathbf{y},\mathbf{z}) \quad = \quad -\langle\, D\mathbf{N}^{\mathrm{local}}(u,v)\mathbf{y}, D\mathbf{X}(u,v)\mathbf{z}\,\rangle$$

where $\langle -,-\rangle$ denotes the usual inner product in $\mathbf{R}^2$. The corresponding classical formula for the First Fundamental Form is

$$\mathbf{I}^{\mathrm{local}}_{(u,v)}(\mathbf{y},\mathbf{z}) \quad = \quad -\langle\, D\mathbf{X}(u,v)\mathbf{y}, D\mathbf{X}(u,v)\mathbf{z}\,\rangle\ .$$

The previous observations yield the following identity for passing from one version of the Second Fundamental Form to the other:

COMPATIBILITY RELATION. *In the preceding discussion, suppose that* $\mathbf{p} = \mathbf{X}(u,v)$, $D\mathbf{X}(u,v)\mathbf{y} = \mathbf{a}$ *and* $D\mathbf{X}(u,v)\mathbf{z} = \mathbf{b}$. *Then we have*

$$\mathbf{II}^{\mathrm{local}}_{(u,v)}(\mathbf{y},\mathbf{z}) \quad = \quad \mathbf{II}^{\Sigma}_{\mathbf{p}}(\mathbf{a},\mathbf{b})\ .■$$

Symmetry property of the Second Fundamental Form

One advantage of the local version of the Second Fundamental Form is that it quickly yields the following basic symmetry property.

PROPOSITION. *In the setting above we have*

$$\mathbf{II}^{\mathrm{local}}_{(u,v)}(\mathbf{y},\mathbf{z}) \quad = \quad \mathbf{II}^{\mathrm{local}}_{(u,v)}(\mathbf{z},\mathbf{y})$$

for all u, v, $\mathbf{y}$ *and* $\mathbf{z}$.

COROLLARY. *In the setting above we have*

$$\mathbf{II}^{\Sigma}_{\mathbf{p}}(\mathbf{a},\mathbf{b}) \quad = \quad \mathbf{II}^{\Sigma}_{\mathbf{p}}(\mathbf{b},\mathbf{a})$$

for all $\mathbf{p}$, $\mathbf{a}$ *and* $\mathbf{b}$.

The corollary follows from the proposition and the compatibility relation between the local and global versions of the Second Fundamental Form.

Proof of Proposition. As in the result at the beginning of Section IV.3, it suffices to prove this when $\mathbf{y}$ and $\mathbf{z}$ are the standard unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. It will be convenient for us to denote the partial derivatives of $\mathbf{N} = \mathbf{N}^{\mathrm{local}}$ and $\mathbf{X}$ by $\mathbf{N}_j$ and $\mathbf{X}_j$ here.

By definition we have

$$\mathbf{II}^{\mathrm{local}}_{(u,v)}(\mathbf{e}_1,\mathbf{e}_2) \quad = \quad -\langle\, \mathbf{N}_1,\, \mathbf{X}_2 \,\rangle$$

and by the computations of Section IV.2 we know that the right hand side is equal to $\langle\, \mathbf{N},\, \mathbf{X}_{2,1} \,\rangle$. Similarly, we have

$$\mathbf{II}^{\mathrm{local}}_{(u,v)}(\mathbf{e}_2,\mathbf{e}_1) \quad = \quad -\langle\, \mathbf{N}_2,\, \mathbf{X}_1 \,\rangle$$

and by the computations of Section IV.2 we know that the right hand side is equal to $\langle\, \mathbf{N},\, \mathbf{X}_{1,2} \,\rangle$. Since $\mathbf{X}_{2,1} = \mathbf{X}_{1,2}$ by equality of mixed partial derivatives, it follows that we have proven the symmetry condition in the special case, and as noted before the general case follows from this.∎

Since the Second Fundamental Form is defined in terms of the standard inner product on $\mathbf{R}^2$ and the shape operator, we also have the following consequence.

PROPOSITION. *In the setting above, for each* $\mathbf{p} \in \Sigma$ *the linear transformaton* $\mathbf{S_p}$ *defined on* $T_{\mathbf{p}}(\Sigma)$ *by the shape operator has the following* SELF – ADJOINTNESS *property:*

$$\langle\, \mathbf{S_p}(\mathbf{a}), \mathbf{b} \,\rangle \quad = \quad \langle\, \mathbf{S_p}(\mathbf{b}), \mathbf{a} \,\rangle$$

Proof. This follows from the previous results because the left and right hand sides are equal to the values of the Second Fundamental Forms at $(\mathbf{a},\mathbf{b})$ and $(\mathbf{b},\mathbf{a})$ respectively.∎

The self-adjointness identity is extremely important, and it is used extensively in the final three sections of the notes.