Line L: all points $\vec{p_0} + t\vec{u}$ t real

M: $\vec{p_1} + s\vec{v}$ s real

$f(s,t)$ = distance squared =

$$|(\vec{p_0} + t\vec{u}) - (\vec{p_1} + s\vec{v})|^2 =$$

$$|\underbrace{(\vec{p_0} - \vec{p_1})}_{=\vec{q}} + (t\vec{u} - s\vec{v})|^2$$

$\longrightarrow$ note $\vec{q}$ is nonzero.
(L & M have no common pts.)

Rewrite with substitution and expand:

$$f(s,t) = |\vec{q}|^2 + 2((t\vec{u} - s\vec{v})\cdot\vec{q}) + |t\vec{u} - s\vec{v}|^2 =$$

$$|\vec{q}|^2 + 2(t\vec{u} - s\vec{v})\cdot\vec{q} + t^2|\vec{u}|^2 - 2st(\vec{u}\cdot\vec{v}) + s^2|\vec{v}|^2.$$

We can test this for relative minima by solving $\frac{\partial f}{\partial s} = 0 = \frac{\partial f}{\partial t}$. However, we need to check there is an <u>absolute</u> minimum.

Intuitively, this is clear but ...

How to see there is an absolute minimum?

Show $f(s,t)$ is large if s^2+t^2 is large.

For this, it is convenient to rewrite $f(s,t)$ using "polar" notation:

$$t = r\cos\theta$$
$$s = r\sin\theta$$

We then get

$f(r\sin\theta, r\cos\theta) =$

$$|\vec{q}|^2 + 2r(\cos\theta\,\vec{u} - \sin\theta\,\vec{v})\cdot\vec{q} + r^2|\cos\theta\,\vec{u} - \sin\theta\,\vec{v}|^2.$$

<u>CLAIM</u> $\lim_{r\to\infty} f(r\cos\theta, r\sin\theta) = \infty$.

Let $\left.\begin{array}{l} m = \text{minimum} \\ M = \text{maximum}\end{array}\right\}$ value of $|\cos\theta\,\vec{u} + \sin\theta\,\vec{v}|$

Note that $m > 0$ because $(\cos\theta\,\vec{u} - \sin\theta\,\vec{v})$ is never zero ($\vec{u}$ + $\vec{v}$ are lin indep.!!)

The Cauchy-Schwarz $\leq$ implies

$$-M\cdot|\vec{q}| \leq (\cos\theta\,\vec{u} - \sin\theta\,\vec{v})\cdot\vec{q} \leq M\cdot|\vec{q}|$$

$$\left(\ |\vec{a}\cdot\vec{b}| \leq |\vec{a}|\cdot|\vec{b}| \quad \text{RECALL}\right)$$

So the second term in the formula for $f(r\sin\theta, r\cos\theta)$ is $\geq -2rM|\vec{q}|$ and the first is $\geq r^2 m^2$.

Hence $f(r\sin\theta, r\cos\theta) \geq$

$$m^2r^2 - 2rM|\vec{q}| + |\vec{q}|^2.$$

The limit of this as $r \to \infty$ is $+\infty$.
(need $m>0$ here)

HENCE

~~We can find $K>0$ so that $f \geq 2K$~~

~~Hence f has a min~~

This means we can find $K>0$ so that

$f \geq 2|\vec{q}|$ if $r \geq K$.

OTOH f has a minimum on the set $s^2+t^2 \leq K$. Since $f(0,0) = |\vec{q}| <$ value of f on the boundary, this minimum is **NOT** on the circle $r = K$.

This means the minimum occurs at a point where $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial t} = 0$.

Where does this happen?

$$0 = \frac{\partial f}{\partial s} = -2\vec{v}\cdot\vec{q} - 2t\,\vec{u}\cdot\vec{v} + 2s\,\underbrace{\vec{v}\cdot\vec{v}}_{=|v|^2}$$

$$0 = \frac{\partial f}{\partial t} = 2\vec{u}\cdot\vec{q} - 2s\,\vec{u}\cdot\vec{v} + 2t\,\underbrace{\vec{u}\cdot\vec{u}}_{=|u|^2}$$

As indicated in the course notes, there is a unique solution ~~of~~ for these equations for s and t.

Call it s^*, t^*.

To finish we need to show that the direction of the line joining

$\vec{p_0} + t^*\vec{u}$ to $\vec{p_1} + s^*\vec{v}$

is perpendicular to both direction vectors $\vec{u}$ and $\vec{v}$.

But $\left[(\vec{p_0} + t^* u) - (p_1 + s^* v)\right] \cdot \vec{u} =$

$\left[\vec{q} + t^*\vec{u} - s^*\vec{v}\right] \cdot \vec{u}$ and this is zero because $\frac{\partial f}{\partial t}(s^*, t^*) = 0$.

notice this is $\frac{1}{2}\frac{\partial f}{\partial t}(s^*, t^*)$.

Likewise $\left[\text{ SAME }\right] \cdot \vec{v} =$

$\left[\vec{q} + t^*\vec{u} - s^*\vec{v}\right] \cdot \vec{v} =$

$-\frac{1}{2}\frac{\partial f}{\partial s}(s^*, t^*) = 0$. So the shortest distance is along a common perpendicular.