Solutions to Examination 2

1. (*a*) By definition we have

$$\cos \angle(A\mathbf{x}, A\mathbf{y}) \quad = \quad \frac{\langle A\mathbf{x}, A\mathbf{y}\rangle}{|A\mathbf{x}|\,|A\mathbf{y}|}$$

and since $A = cB$ where $c > 0$ and B is orthogonal we have

$$\frac{\langle A\mathbf{x}, A\mathbf{y}\rangle}{|A\mathbf{x}|\,|A\mathbf{y}|} \quad = \quad \frac{\langle cB\mathbf{x}, cB\mathbf{y}\rangle}{|cB\mathbf{x}|\,|cB\mathbf{y}|} \quad =$$

$$\frac{\langle c^2 B\mathbf{x}, B\mathbf{y}\rangle}{|c|^2\,|B\mathbf{x}|\,|B\mathbf{y}|} \quad = \quad \frac{\langle B\mathbf{x}, B\mathbf{y}\rangle}{|B\mathbf{x}|\,|B\mathbf{y}|}\ .$$

Since orthogonal matrices preserve lengths and inner products, it follows that the right hand side is equal to

$$\frac{\langle \mathbf{x}, \mathbf{y}\rangle}{|\mathbf{x}|\,|\mathbf{y}|} \quad = \quad \cos \angle(\mathbf{x}, \mathbf{y})$$

which is what we needed to prove.■

(*b*) For each j, the j^{th} column of A is the vector $A\mathbf{e}_j$, where $\mathbf{e}_j$ denotes the standard j^{th} unit vector. Since $\cos \angle(\mathbf{e}_i, \mathbf{e}_j) = 0$ if $i \neq j$, and since A is conformal it follows that

$$0 \quad = \quad \cos \angle(\mathbf{e}_i, \mathbf{e}_j) \quad = \quad \cos \angle(A\mathbf{e}_i, A\mathbf{e}_j)$$

if $i \neq j$.■

(*c*) Following the hint, if $i \neq 1$ then we have

$$\cos \angle(\mathbf{e}_1 + \mathbf{e}_i, \mathbf{e}_1) \quad = \quad \frac{\langle \mathbf{e}_1 + \mathbf{e}_i, \mathbf{e}_1\rangle}{|\mathbf{e}_1 + \mathbf{e}_i|\,|\mathbf{e}_1|} \quad = \quad \frac{1}{\sqrt{2}}$$

and since A is conformal we have the same value for $\cos \angle(\,A(\mathbf{e}_1 + \mathbf{e})_i, A\mathbf{e}_1)$. Since $\mathbf{e}_i$ and $\mathbf{e}_1$ are perpendicular and similarly for their images under A, we have the following:

$$\langle\, A(\mathbf{e}_1 + \mathbf{e}_i)), A\mathbf{e}_1\,\rangle \quad = \quad L_1^2$$

$$|A(\mathbf{e}_1 + \mathbf{e}_i)|^2 \quad = \quad |A\mathbf{e}_1|^2 \ + \ |A\mathbf{e}_i|^2 \quad = \quad L_1^2 \ + \ L_i^2$$

If we combine this with the previous observations we see that

$$\frac{1}{\sqrt{2}} \quad = \quad \cos \angle\big(\,A(\mathbf{e}_1 + \mathbf{e}_i), A\mathbf{e}_1\,\big) \quad = \quad \frac{L_1^2}{L_1\sqrt{L_1^2 + L_i^2}} \quad = \quad \frac{L_1}{\sqrt{L_1^2 + L_i^2}}$$

and clearing the latter of fractions yields the equation $\sqrt{L_1^2 + L_i^2} = L_1\sqrt{2}$. Since L_1 and L_i are positive, the only solution to this equation is $L_i = L_1$.■

(*d*) By (*c*) we know that $A\mathbf{e}_i = L_1\mathbf{e}_i$ for all i. Thus if $c = L_1$ it follows that $B = c^{-1}A$ is a matrix which sends the standard unit vectors to an orthonormal set, and therefore B must be orthogonal. Since $A = cB$, we have the required conclusion.■

(e) The matrix $Df(u,v)$ is equal to

$$\begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}$$

and if we put this into polar coordinates as suggested we obtain the matrix

$$\begin{pmatrix} 2r\cos\theta & -2r\sin\theta \\ 2r\sin\theta & 2r\cos\theta \end{pmatrix} \quad = \quad 2r \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

so that by (a) the matrix is conformal if $r \neq 0$, which is equivalent to $(u,v) \neq (0,0)$.■

2. (a) In order to show we have a surface, it is necessary to verify that if $f(x,y,z) = 0$ then $\nabla f(x,y,z) \neq 0$. If $f(x,y,z) = 0$, then $xyz = 1$, and as noted in the hint this means that all three coordinates must be nonzero; this is true because the contrapositive statement — if one factor is zero, then the whole product is zero — is elementary to check. But now $\nabla f(x,y,z) = (yz, zx, xy)$ and if all three of x, y, z are nonzero then all three coordinates of the gradient are also nonzero. In particular, this holds if $xyz = 1$.■

(b) Since the normal direction to the tangent plane at a point of V_f is given by the gradient, we need to find all (x,y,z) such that $\nabla f(x,y,z)$ is a multiple of (a,b,c). By the preceding part of the problem, we need to find all $(x,y,z) \in V_f$ such that

$$\nabla f(x,y,z) \quad = \quad (yz, zx, xy) \quad = \quad \lambda(a,b,c)$$

for some nonzero scalar λ. If we take the product of the coordinates on both sides of this equation, we obtain the following:

$$1 \quad = \quad (xyz)^2 \quad = \quad (yz)(zx)(xy) \quad = \quad \lambda^3 abc$$

which yields $\lambda = (abc)^{-1/3}$. Let $a' = \lambda a$, $b' = \lambda b$, and $c' = \lambda c$. Then we are reduced to solving the system of equations given by

$$(yz, zx, xy) \quad = \quad (a', b', c')$$

and these have a unique solution given as follows:

$$x \;=\; \left(\frac{b'c'}{(a')^2}\right)^{1/3} \;=\; \left(\frac{bc}{a^2}\right)^{1/3} \qquad y \;=\; \left(\frac{a'c'}{(b')^2}\right)^{1/3} \;=\; \left(\frac{ac}{b^2}\right)^{1/3} \qquad z \;=\; \left(\frac{a'b'}{(c')^2}\right)^{1/3} \;=\; \left(\frac{ab}{c^2}\right)^{1/3}$$

Therefore for each (a,b,c) there is exactly one point on V_f such that the tangent plane's normal direction is given by (a,b,c). Note that each of x, y, z is positive.

The preceding shows that the tangent plane to the surface at the given point is either equal or parallel to the original plane, which passes through the origin. To see that the tangent plane must be parallel and not equal to this plane, all we have to do is observe that the original plane is defined by the equation $au + bv + cw = 0$ and for the positive numbers x, y, z found in the previous paragraph we have $ax + by + cz$ is a sum of products of positive numbers and hence is also positive, so that (x,y,z), which lies on the tangent plane by definition, cannot also lie on the original plane.■

3. (a) The partial derivatives of $\mathbf{X}(u,v) = \mathbf{a}(u) + v(\mathbf{e}_3 - \mathbf{a}(u))$ are $\mathbf{X}_1 = (1-v)\mathbf{a}$ and $\mathbf{X}_2 = \mathbf{e}_3 - \mathbf{a}$. Therefore the factors of the cross product $\mathbf{X}_1 \times \mathbf{X}_2$ are such that $\mathbf{X}_1$ is a nonzero vector in the xy-plane and $\mathbf{X}_2$ has a nonzero third coordinate. There is no way that either of these nonzero vectors can be a multiple of the other, so they must be linearly independent and consequently their cross product must be nonzero.■

(*b*) By the definitions we have

$$\mathbf{X}_1 \times \mathbf{X}_2(u,v) \quad = \quad (1-v)\mathbf{a}'(u) \times \big(\mathbf{e}_3 - \mathbf{a}(u)\big)$$

and therefore at each point of the ruling $u = \mathsf{CONSTANT}$ the unit normal is the same. The ruling L through $\mathbf{X}(u,v)$ is perpendicular to the unit normal (to both the surface and the tangent plane!) and therefore the line containing it lies in the tangent plane $P(u,v)$ to the surface at $\mathbf{X}(u,v)$.■

4. (*a*) Write $f(\mathbf{v}) = A\mathbf{v} + \mathbf{b}$ as suggested, where A is an orthogonal matrix and $\mathbf{b}$ is some vector. The tangent plane Y to the original surface at $\mathbf{p}$ is the unique plane through $\mathbf{p} = \mathbf{X}(u,v)$ whose normal direction is given by $\mathbf{X}_1(u,v) \times \mathbf{X}_2(u,v)$. Similarly, the tangent plane Z to the image surface at $f(\mathbf{p})$ is the unique plane through $f(\mathbf{p}) = f \circ \mathbf{X}(u,v)$ whose normal direction is given by $[f \circ \mathbf{X}]_1(u,v) \times [f \circ \mathbf{X}]_2(u,v)$. By the Chain Rule, the latter is equal to $A\mathbf{X}_1(u,v) \times A\mathbf{X}_2(u,v)$. If $\det A = +1$ then we know that this vector is equal to $A(\,\mathbf{X}_1(u,v) \times \mathbf{X}_2(u,v)\,)$. On the other hand, if $\det A = -1$ then $\det -A = +1$ and we have

$$-A\big(\mathbf{X}_1(u,v) \times \mathbf{X}_2(u,v)\big) \quad = \quad -A(\,\mathbf{X}_1(u,v)\,) \times -A(\,\mathbf{X}_2(u,v)\,) \quad = \quad A(\,\mathbf{X}_1(u,v)\,) \times A(\,\mathbf{X}_2(u,v)\,)$$

and hence in all cases we have

$$A\big(\mathbf{X}_1(u,v) \times \mathbf{X}_2(u,v)\big) \quad = \quad \det A \cdot \Big(A(\,\mathbf{X}_1(u,v)\,) \times A(\,\mathbf{X}_2(u,v)\,)\Big)$$

so that if $\mathbf{N}$ is a normal direction for the original surface and its tangent plane Y at $\mathbf{p}$, then $A\mathbf{N}$ is a normal direction for the tangent plane Z to the image surface at $f(\mathbf{p})$; the key point for our purposes is that $\pm A\mathbf{N}$ determine the same normal line. — On the other hand, the unique plane through $f(\mathbf{P})$ with normal direction $A\mathbf{N}$ is also given by the image plane $f(Y)$, and therefore it follows that $Z = f(Y)$ as required.■

ALTERNATE APPROACH. Here is another way of working the previous exercise that applies more generally to an arbitrary **affine mapping** $f(\mathbf{v}) = A\mathbf{v} + \mathbf{b}$ where A is invertible and $\mathbf{b}$ is arbitrary. It is a straightforward exercise to show that every such map has a global inverse that is also an affine mapping, but we shall not verify this here (you should be able to solve the equation $\mathbf{w} = A\mathbf{v} + \mathbf{b}$ for $\mathbf{v}$ directly in terms of $\mathbf{b}$ and A^{-1}). — One characterization of the tangent plane Y is that it consists of all vectors of the form $\mathbf{p} + \mathbf{v}$, where $\mathbf{v}$ is the tangent vector to some smooth curve γ in the surface at the point $\mathbf{p}$. By the Chain Rule, if $f \circ \gamma$ is the image of this curve under f, then the tangent vector to the image curve $f \circ \gamma$ at the image point $f(\mathbf{p})$ is just $A\mathbf{v}$. Therefore the tangent plane to the image surface at $f(\mathbf{p})$ is a plane containing all vectors of the form $f(\mathbf{p}) + A\mathbf{v}$, where $\mathbf{v}$ is as before. By the definition of f we know that $f(\mathbf{p}) + A\mathbf{v} = f(\mathbf{p} + \mathbf{v})$, so the tangent plane at the image point contains $f(\mathbf{y})$ for all $\mathbf{y}$ in the tangent plane at $\mathbf{p}$. Since the tangent plane at a point $\mathbf{q}$ consists of all vectors of the form $\mathbf{q} + \mathbf{w}$ for all $\mathbf{w}$ in some 2-dimensional subspace W, it follows that the tangent plane at $f(\mathbf{p})$ must be precisely the image of the tangent plane at $\mathbf{p}$ under the mapping f (in more detail, we have two 2-dimensional subspaces W_1 and W_2 and a fixed vector $\mathbf{q}$ such that the set of all vectors of the form $\mathbf{q} + \mathbf{y}$ with $\mathbf{y} \in W_1$ is contained in the set of all vectors $\mathbf{q} + \mathbf{z}$ with $\mathbf{z} \in W_2$. If this happens then W_1 must be contained in W_2, and since the dimensions of the two subspaces are equal, it follows from elementary linear algebra that the subspaces themselves must also be equal).■

(*b*) Let Y and Z be as above. In this case Z is a plane through $h(\mathbf{p}) = c\mathbf{p}$ with the same normal direction as Y because $h \circ \mathbf{X}_1 \times h \circ \mathbf{X}_2 = c\mathbf{X}_1 \times c\mathbf{X}_2 = c^2(\mathbf{X}_1 \times \mathbf{X}_2)$. Now the plane cY, consisting of all vectors $c\mathbf{w}$ for $\mathbf{w} \in Y$, also contains $h(\mathbf{p})$ and as the same direction as the other plances, and thus by uniqueness we see that $Z = cY$.■

(c) The intersection $P \cap Q$ is a line if and only if the normal directions for the two planes are different. Since the normal directions for Y and Z are the same, their intersection cannot be equal to a line. The other possibilities are easy fo realize. For the unit sphere. the tangent planes Y and Z are parallel; it is only necessary to check that one contains a point that is not in the other. But if $\mathbf{p} = 1$, then Y is defined by $(\mathbf{y} - \mathbf{p}) \cdot \mathbf{p} = 0$ and since $(c\mathbf{p} - \mathbf{p}) \cdot \mathbf{p} = c - 1 \neq 0$ it follows that $c\mathbf{p} = h(\mathbf{p}) \notin Y$. To realize the possibility $Y = Z$, take the surface $z = 0$ (the xy-plane). Then h maps this plane to itself and for each $\mathbf{p}$ the tangent planes Y and Z are also equal to the xy-plane. Alternatively, on can realize the second possibility using the sphere defined by the equation $(x - 1)^2 + y^2 + z^2 = 1$ and taking $\mathbf{p}$ to be the origin, so that the tangent planes to both the original surface and its image at $\mathbf{p} = h(\mathbf{p})$ are both given by the xy-plane.■