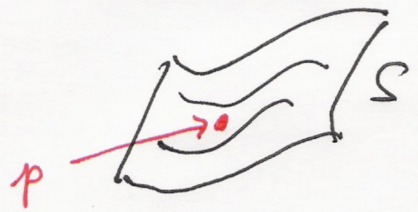
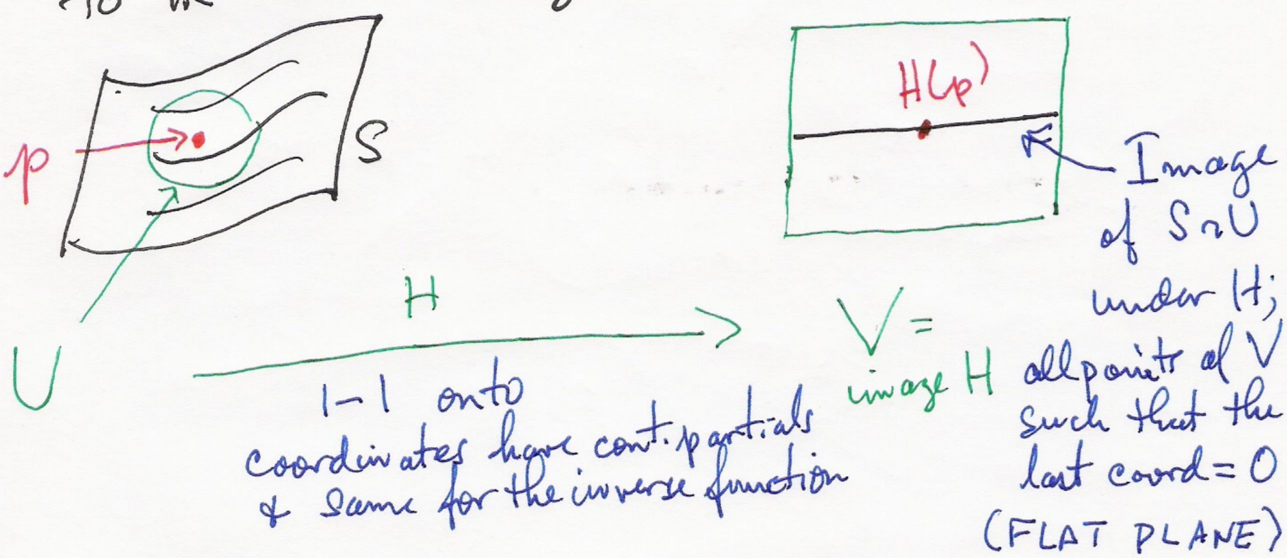
Geometric Surfaces

$S \subseteq \mathbb{R}^3$ subset



The defining condition is that if p lies on S then we can find an open set U containing p and a change of variables map H from U to $\mathbb{R}^3$ which locally "flattens out" the surface:



<u>Question</u>: How does this relate to more concrete descriptions of surfaces?

<u>Parametrized surfaces:</u> U open in $\mathbb{R}^2$

$p \in U$, $\vec{X} : U \longrightarrow \mathbb{R}^3$ regular parametrization:

$$\frac{\partial \vec{X}}{\partial u} \times \frac{\partial \vec{X}}{\partial v} \neq \vec{0} \text{ everywhere}$$

$$\Phi(u, v, w) = \vec{X}(u,v) + w\left(\vec{X}_1(u,v) \times \vec{X}_2(u,v)\right)$$

$\vec{X}_i$ = partial derivative w.r.t. ith variable.

<u>CLAIM</u> $D\Phi(u, v, 0)$ is invertible

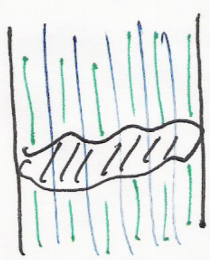
To see this write out the matrix explicitly:

$$\left(\vec{X}_1 \;\vdots\; \vec{X}_2 \;\vdots\; \vec{X}_1 \times \vec{X}_2 \right)$$

The determinant of this is $|\vec{X}_1 \times \vec{X}_2|^2$, which is nonzero because $\vec{X}_1 \times \vec{X}_2 \neq \vec{0}$.

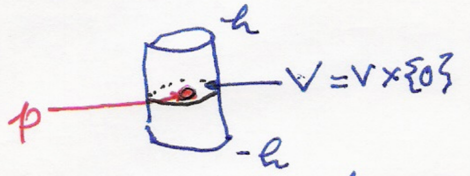
Let $U^* = U \times \mathbb{R}$ = all (u,v,w) such that $(u,v) \in U$.

$$\boxed{\text{Φ is defined on the open set U^*}}$$



$\longleftarrow$ U, identified with $U \times \{0\}$ = all points of the form $(u, v, 0)$.

By the Inverse Function Theorem, there is an open set of the form $V \times (-h, h)$ in U^* s.t. V is open in $\mathbb{R}^2$, $p \in V$



$(u, v) \in V$ and
$-h < w < h$

and Φ maps $V \times (-h, h)$ 1-1 onto an open set W containing $\vec{X}(p) = \Phi(p, 0)$.

Let H be an inverse map from W to the set $V \times (-h, h)$. Then under H the set $(\text{Image } \vec{X} \cap W)$ corresponds to $V \times \{0\}$.

Therefore, if $\vec{X}$ is a regular parametrization and $p \in U$ (the domain of $\vec{X}$), then one can find an open set V such that $p \in V \subseteq U$ and the image $\vec{X}[V]$ is a geometric surface as defined in the notes.

An example near the end of III.2 shows that sometimes one cannot avoid restricting $\vec{X}$ to smaller open subsets of $\mathbb{R}^2$.

<u>Level surfaces:</u> To simplify the discussion, let's add or subtract a constant so that the object of interest is defined by an equation $F(x,y,z) = 0$, where F has continuous partial derivatives and is defined on some open set U in $\mathbb{R}^3$.

The crucial regularity condition is the same one which arises in the discussion of Lagrange multipliers:

$$\boxed{\text{If } F(x_0, y_0, z_0) = 0, \text{ then } \nabla F(x_0, y_0, z_0) \neq \vec{0}}$$

"Quadric surfaces" in $\mathbb{R}^3$ provide an important family of examples; in the case of cones $x^2+y^2-z^2=0$ one must remove the origin from the open set U. The most important example within this family is probably the sphere with center $\vec{0}$ and radius 1, which is defined by $0 = x^2+y^2+z^2-1$.

We need to check that if $V(F)$ is the zero set of F, where F satisfies the preceding regularity condition, then $V(F)$ satisfies the condition for a geometric surface.

We shall only consider the case where $p = (x_0, y_0, z_0)$ and $\frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0$. The cases where $\frac{\partial F}{\partial x}(x_0, y_0, z_0) \neq 0$ or $\frac{\partial F}{\partial y}(x_0, y_0, z_0) \neq 0$ can be handled similarly by interchanging the roles of the x, y and z coordinates.

Define $\Phi : U \longrightarrow \mathbb{R}^3$ by

$$\Phi(x, y, z) = (x, y, F(x,y,z)).$$

<u>CLAIM</u>: $D\Phi(x_0, y_0, z_0)$ is invertible.

If we write out this matrix, we get the following:

$$\left(\begin{array}{c:c:c} 1 & 0 & \\ 0 & 1 & \nabla F(x_0, y_0, z_0) \\ 0 & 0 & \end{array} \right)$$

The determinant of this matrix is $\frac{\partial F}{\partial z}(x_0, y_0, z_0)$, which is nonzero.

Again by the Inverse Function Theorem, there is an open set V^* such that $p \in V \subseteq U$ and Φ maps V^* in a 1-1 onto fashion to an open set W containing $\Phi(p) = (x_0, y_0, 0)$. In fact, one also has a good inverse h from W to V^*, just as before.

Under the mapping Φ, the piece of the surface given by $V(F) \cap V^*$ corresponds to the set of points in W for which the third coordinate equals zero.

Therefore, if F is as above, then it follows that $V(F)$ is a geometric surface.

Once again, if there are points (x, y, z) such that $F(x, y, z) = 0$ and $\nabla \vec{F}(x, y, z) = \vec{0}$, then $V(F)$ may fail to satisfy the criterion for a surface at these <u>singular points</u>.