Finding local parametrizations

We have noted that sufficiently small pieces of parametrized surfaces are geometric surfaces. There is also a converse:

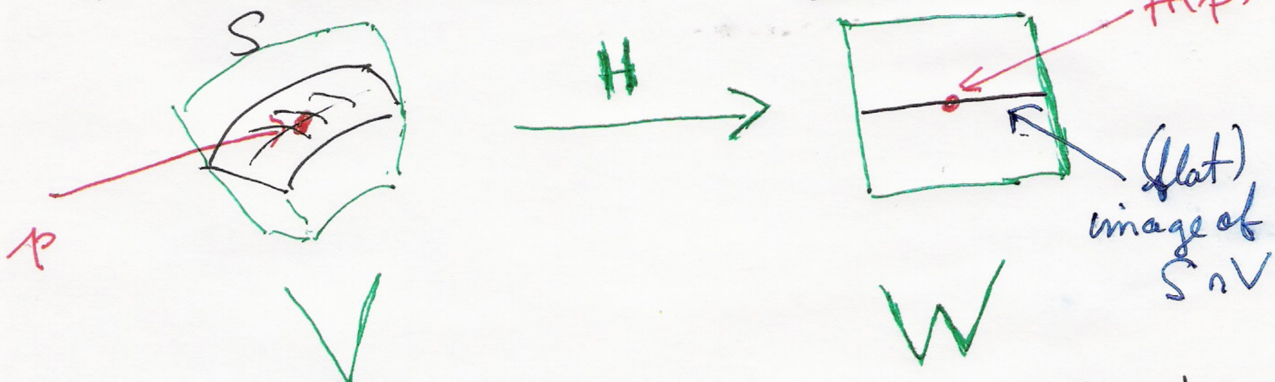
<u>**CLAIM:**</u> If $S \subseteq \mathbb{R}^3$ is a geometric surface and $p \in S$, then there is an open set V containing p such that $V \cap S$ is the image of a ~~[illegible]~~ regular parametrization $\vec{\sigma}$.

Often $\vec{\sigma}$ is called a <u>local coordinate patch</u> at the point p.

We have already constructed local coordinate patches for all points on the sphere defined by $x^2+y^2+z^2-1=0$.

It is fairly straightforward to verify the claim. By definition, there is an open set V containing p and a "flattening map" $H: V \longrightarrow W$ such that H is 1-1 onto an open set W, both H and its inverse have coordinate functions with continuous partial derivatives, and H sends $V \cap S$ to the set of all (x, y, z) in W such that $z = 0$.

$\boxed{\text{Also the Jacobians of } H \text{ \& } H^{-1} \text{ are never zero}}$



If, as usual, H^{-1} is the inverse function for H, then we can take $\vec{\sigma}(u,v) = H^{-1}(u, v, 0)$.

By construction, the coordinate functions for $\vec{\sigma}$ have continuous partial derivatives, so it is only necessary to check that $\frac{\partial \vec{\sigma}}{\partial u} \times \frac{\partial \vec{\sigma}}{\partial v} \neq \vec{0}$ everywhere. Since $\frac{\partial \vec{\sigma}}{\partial u}$ & $\frac{\partial \vec{\sigma}}{\partial v}$ form a ~~basis~~ spanning set for the image of $D\vec{\sigma}(u,v)$, which is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$, the cross product condition, which is equivalent to the linear independence of $\frac{\partial \vec{\sigma}}{\partial u}$ and $\frac{\partial \vec{\sigma}}{\partial v}$, is equivalent to the statement that the rank of $D\vec{\sigma}(u,v)$ is equal to 2.

The preceding statement can be verified as follows: By construction $\vec{\sigma}(u,v) = H^{-1}(u,v,0)$. Therefore we have

$$D\sigma(u,v)\begin{pmatrix}1\\0\end{pmatrix} = DH^{-1}(u,v,0)\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$D\sigma(u,v)\begin{pmatrix}0\\1\end{pmatrix} = DH^{-1}(u,v,0)\begin{pmatrix}0\\1\\0\end{pmatrix}$$

So everything reduces to checking that the two vectors on the right hand side are linearly independent. However, this is certainly true because the Jacobian of $DH^{-1}(u,v,w)$ is always nonzero so that $DH^{-1}(u,v,w)$ is always invertible and hence sends each ~~sets~~ set of linearly independent vectors into a[nother] set of linearly independent vectors.