TAKE HOME ANSWER KEY

$$\boxed{F_t = \frac{\partial F}{\partial t}}$$

1. $\omega = P\,dx, \quad \theta = Q\,dy$

Then $\omega \wedge \theta = PQ\,dx \wedge dy$

$$d(\omega \wedge \theta) = d(PQ) \wedge dx \wedge dy =$$
$$\left[(PQ)_x\,dx + (PQ)_y\,dy + (PQ)_z\,dz\right] \wedge dx \wedge dy =$$
$$(PQ)_z\,dz \wedge dx \wedge dy = (PQ)_z\,dx \wedge dy \wedge dz =$$
$$(P_z Q + P Q_z) \cdot dx \wedge dy \wedge dz.$$

Also, $(d\omega) \wedge \theta = (dP) \wedge dx \wedge (Q\,dy)$
$$(P_x\,dx + P_y\,dy + P_z\,dz) \wedge dx \wedge (Q\,dy) =$$
$$(P_z Q)\,dz \wedge dx \wedge dy = (P_z Q)\,dx \wedge dy \wedge dz$$

and furthermore

$$\omega \wedge d\theta = (P\,dx) \wedge (dQ) \wedge dy =$$
$$(P\,dx) \wedge (Q_x\,dx + Q_y\,dy + Q_z\,dz) \wedge dy =$$
$$(PQ_z)\,dx \wedge dz \wedge dy = -PQ_z\,dx \wedge dy \wedge dz.$$

Therefore we also have

$$-\omega \wedge d\theta = P Q_z \, dx \wedge dy \wedge dz$$

So that

$$(d\omega) \wedge \theta - \omega \wedge (d\theta) = (P_z Q + P Q_z) dx \wedge dy \wedge dz$$

and we know the right hand side is equal to $d(\omega \wedge \theta)$.

<u>Note</u>: The identity in the problem is valid for all 1-forms ω and θ. By the Distributive Identity for the wedge product, it is enough to check this when ω has the form $P\,dx$, $Q\,dy$ or $R\,dz$ and θ has the form $L\,dx$, $M\,dy$ or $N\,dz$. There are 9 different possibilities, but up to interchanging the roles of x, y, z ~~the~~ it ~~choices can be~~ suffices to consider the case in the exercise and the case $\omega = P\,dx$, $\theta = L\,dx$ (in which case $\omega \wedge \theta = 0$).

2. Let $g(x,y) = x f\left(\frac{y}{x}\right)$

So the surface is given by $\left(u, v, u f\left(\frac{v}{u}\right)\right)$ and a normal field by $\left(-\frac{\partial g}{\partial u}, -\frac{\partial g}{\partial v}, 1\right)$.

Now $\frac{\partial g}{\partial u} = \frac{\partial}{\partial u}\left(u \cdot f\left(\frac{v}{u}\right)\right) = f\left(\frac{v}{u}\right) - \frac{vu}{u^2} f'\left(\frac{v}{u}\right)$

$$\frac{\partial g}{\partial v} = \frac{1}{u} \cdot u f'\left(\frac{v}{u}\right) = f'\left(\frac{v}{u}\right).$$

Hence the equation of the tangent plane at $(a, b, g(a,b))$ is given by

$$\left[f\left(\tfrac{b}{a}\right) - \tfrac{ba}{a^2} f'\left(\tfrac{b}{a}\right)\right] x + f'\left(\tfrac{b}{a}\right) y - z =$$

$$\left[f\left(\tfrac{b}{a}\right) - \tfrac{b}{a} f'\left(\tfrac{b}{a}\right)\right] a + f'\left(\tfrac{b}{a}\right) \cdot b - a f\left(\tfrac{b}{a}\right)$$

and the right side simplifies to zero.

Hence the tangent plane equations have the form $Px + Qy - z = 0$, and from this we see that $(0, 0, 0)$ lies on each of them.

3. (a) $\vec{X}(v,z) = (z\cos v, z\sin v, z)$

Compute partial derivatives

$$\vec{X}_1 = (-z\sin v, z\cos v, 0)$$

$$\vec{X}_2 = (\cos v, \sin v, 1).$$

$$E = \vec{X}_1 \cdot \vec{X}_1 = z^2 \quad G = \vec{X}_2 \cdot \vec{X}_2 = 2$$

$$F = \vec{X}_1 \cdot \vec{X}_2 = 0$$

and FFF $= z^2 dv dv + 2\, dz dz$.

(b) $\vec{X}_1 = (\cos\theta, \sin\theta, 0)$

$$\vec{X}_2 = (-r\sin\theta, r\cos\theta, 1).$$

$$E = 1, \quad G = r^2+1, \quad F = 0$$

FFF $= dr dr + (r^2+1) d\theta d\theta.$

4. Use the Jacobian test:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2u-v & -1 & 0 \\ -u & 1 & -w \\ 0 & 0 & 2w-v \end{vmatrix}$$

We need to evaluate this at $(1,1,1)$ in (a) and $(1,-1,1)$ in (b).

(a) The Jacobian is $\begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 0.$

(b) The Jacobian is $\begin{vmatrix} 3 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 3 \end{vmatrix} = 6.$

Since the Jacobian in (b) is nonzero, the Inverse Function Theorem implies that (b) must be true, and by process of elimination (a) must be false.

A more explicit way of checking that ($\underline{a}$) is false is to notice that for all real numbers t we have $f(t,t,t) = (0,0,0)$ so clearly there is no unique solution to $f(u,v,w) = (0,0,0)$. Similarly, we have $f(t,t,s) = (0,0,s^2-st)$, and if $z \neq 0$ this means that $(0,0,z) =$

$$f\left(t, t, \tfrac{1}{2}\left(t \pm \sqrt{t^2+z^2}\right)\right)$$

THIS IS POSITIVE SINCE $z \neq 0$.