NOTE ON THE TAKE HOME ASSIGNMENT

Everyone correctly showed that the Jacobians in (a) and (b) were zero and nonzero respectively. This implies that (b) must be true by the Inverse Function Theorem, but the vanishing of the Jacobian by itself is not enough to conclude that (a) is false; an example is given in the next paragraph. However, since we were given that **one** of the statements was true and we knew that (b) was true, we could conclude that (a) cannot be true by process of elimination. As noted on the final page of the answer key, one can also show **directly** that (a) is not true by checking that the relevant systems of equations do not always have unique solutions close to the prescribed values.

As noted above, the point is that even if one has a vanishing Jacobian for $\mathbf{g}(u, v, w)$ at (u_0, v_0, w_0) and $(a, b, c) = \mathbf{g}(u_0, v_0, w_0)$, it might still be possible to find unique solutions to the systems $(x, y, z) = \mathbf{g}(u, v, w)$ for all (x, y, z) sufficiently close to (a, b, c). For example consider $\mathbf{g}(u, v, w) = (u^3, v^3, w^3)$. Then the Jacobian is zero when $(u, v, w) = (0, 0, 0)$, but for every (x, y, z) there is a unique solution to the system $(x, y, z) = \mathbf{g}(u, v, w)$; specifically, take u, v, w to be the unique real cube roots of x, y, z respectively.

Moral. Suppose that the question had been varied slightly to involve the choice of $\mathbf{g}$ considered above, and also suppose it asked for which of the values $(0, 0, 0)$ and $(1, 1, 1)$ there were unique solutions of $(x, y, z) = \mathbf{g}(u, v, w)$ near the given values. The Jacobian test would imply a positive answer for $(1, 1, 1)$ but would **yield no information** for $(0, 0, 0)$; this is all one can say using only information about the Jacobian. In fact, for this example one still has unique solutions despite the vanishing of the Jacobian, so it would be incorrect to claim that the Jacobian computation shows that unique solutions do not exist. near $(0, 0, 0)$, for in fact they do exist.