EXERCISES FOR MATHEMATICS 138A

WINTER 2004

The references denote sections of the text for the course:

M. P. do Carmo, *Differential geometry of Curves and Surfaces*, Prentice-Hall, Saddle River NJ, 1976, ISBN 0–132–12589–7.

I. Classical Differential Geometry of Curves

I.1 : Cross products

(O'Neill, § 2.2)

Additional exercise

1. Verify that the cross product of vectors in $\mathbf{R}^3$ satisfies the *Jacobi identity*:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \; + \; \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) \; + \; \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \quad = \quad \mathbf{0} \; .$$

I.2 : Parametrized curves

(O'Neill, § 1.4)

O'Neill, pp.21-22: 2, 8

Additional exercises

1. Find a parametrized curve $\alpha(t)$ which traces out the unit circle about the origin in the coordinate plane and has initial point $\alpha(0) = 1$.

2. Let $\alpha(t)$ be a parametrized cure which does not pass through the origin. If $\alpha(t_0)$ is the point in the image that is closest to the origin and $\alpha'(t_0) \neq 0$, show that $\alpha(t_0)$ and $\alpha'(t_0)$ are perpendicular.

3. Two lines are said to be *skew lines* if they are disjoint but not parallel. Prove that the distance between the skew lines $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{u}$ and $\mathbf{y}(t) = \mathbf{y}_0 + t\mathbf{v}$ is given by

$$\rho \quad = \quad \frac{\mathbf{u} \times \mathbf{v} \cdot \mathbf{r}}{\mathbf{u} \times \mathbf{v}}$$

where $\mathbf{r} = \mathbf{x}_0 - \mathbf{y}_0$. [*Hints:* The shortest distance between the lines is given by a common perpendicular. You may assume the existence of a common perpendicular when working the problem. It might be helpful to let $\mathbf{x}_1$ and $\mathbf{y}_1$ from these lines lie on this common perpendicular.]

4. Prove that a regular smooth curve lies on a straight line if and only if there is a point that lies on all its tangent lines.

I.3 : Arc length and reparametrization

(O'Neill, §§ 1.4, 2.2)

O'Neill, pp. 56–57: 3–5, 10, 11

Additional exercises

1. Prove that a necessary and sufficient condition for the plane $\mathbf{N} \cdot \mathbf{x} = 0$ to be parallel to the line $\mathbf{x} = \mathbf{x}_0 + t \cdot \mathbf{u}$ is for $\mathbf{N}$ and $\mathbf{u}$ to be perpendicular.

2. (a) Given $a > 0$, consider the set of all continuously differentiable real valued functions f on $[0,1]$ such that $f(0) = 0$ and $f(1) = a > 0$. Define $L(f)$ by the formula $L(f) \;=\; \int_0^a |f'(t)|\, dt$. Show that the minimum value of $L(f)$ is a, and if equality holds then f' is everywhere nonnegative. [*Hints:* Since $f' \leq |f'|$ a similar inequality holds for their definite integrals. This inequality of integrals is strict if and only if $f'(t) < |f'(t)|$ for some t, which happens if and only if $f'(t) < 0$ for that choice of t.]

(b) Let ρ, θ and ϕ denote the usual spherical coordinates, and suppose we have a curve on the sphere of radius 1 about the origin with parametric equations of the form

$$\mathbf{x}(t) \;=\; (\cos\theta(t)\, \sin\phi(t),\, \sin\theta(t)\, \sin\phi(t),\, \cos\phi(t)\,)$$

for continuously differentiable functions $\theta(t)$ and $\phi(t)$. Prove that the length of this curve is given by the formula

$$\int_a^b \sqrt{(\theta'(t))^2 \;+\; \sin^2\theta(t)\,(\phi\,'(t))^2}\, dt$$

where the curve is defined on $[a, b]$.

(c) Show that among all regular smooth curves $\mathbf{x}$ that are defined on $[0,1]$, have images on the unit spere, and connect the points $(1,0,0)$ and $(\cos a, \sin a, 0)$ for some $a < \pi$, the curve of shortest length is given by the great circle arc joining the endpoints, and that any other curve with this length is a weak reparametrization of the great circle arc (*i.e.*, if α is the standard great circle arc, then any other curve β must have the form $\beta(t) = \alpha(\,f(t)\,)$, where f is a 1–1 function from $[0,1]$ to $[0,a]$ that is continuously differentiable and satisfies $f' \geq 0$. [*Hints:* Let $\mathbf{y}$ be the curve in the xy-plane obtained from $\mathbf{x}$ by replacing $\phi(t)$ with $\pi/2$; in other words, $\mathbf{y}$ is the perpendicular projection of the original curve onto the xy-plane. Why does the spherical coordinate arc length formula show that the length of $\mathbf{x}$ is greater than or equal to the length of $\mathbf{y}$? And why is there strict inequality if $\phi\,'(t_0)\,\sin\theta(t_0) \neq 0$ somewhere? Why does this mean that the plane curve $(cos\theta(t), \sin\theta(t), 0)$ is a weak reparametrization of $(\cos a\,t, \sin a\,t, 0$? Recall that by continuity the latter implies $\phi\,'(t) \neq 0$ for all t sufficiently close to t_0. What does part (a) imply if ϕ is constant?]

Note. The final part of the problem is a special case of the well known result that the shortest curve on a sphere joining two points is given by the smaller of the arcs on the great circle through the points; in fact, one can use this special case to prove the general statement. [A file containing a detailed proof may be inserted into the course directory eventually.]

I.4 : Curvature and torsion

(O'Neill, § 2.3)

Additional exercises

1. Suppose a curve is given in polar coordinates by $r = r(\theta)$ where $\theta \in [a, b]$.
(*i*) Show that the arc length is $\int_a^b \sqrt{r^2 + (r')^2}\, d\theta$.
(*ii*) Show that the curvature is

$$k(\theta) = \frac{2(r')^2 - rr' + r^2}{[r^2 + (r')^2]^{3/2}} .$$

2. Let α and β be regular parametrized curves such that β is the arc length reparametrization of α. Let t be the parameter for α and s for β. Prove the following:

(*a*) $dt/ds = 1/|\alpha'|$, $d^2t/ds^2 = -(\alpha' \cdot \alpha''/|\alpha'|^4$

(*b*) The curvature is given by

$$k(t) = \frac{\alpha' \times \alpha''}{|\alpha'|^3}$$

(*c*) The torsion is given by

$$\tau(t) = -\frac{\alpha' \times \alpha'' \cdot \alpha'''}{|\alpha' \times \alpha''|^2}$$

(*d*) If α has coordinate functions x and y, then the signed curvature of α at t is equal to

$$k(t) = \frac{x'y'' - x''y'}{[(x')^2 + (y')^2]^{3/2}}$$

3. Show that the curvature of a regular parametrized curve α at t_0 is equal to the curvature of the plane curve γ which is the perpendicular projection of α onto the osculating plane of α at t_0.

4. Consider the problem of designing a set of railroad tracks that contains a pair of parallel tracks along with a third going from the first to the second smoothly. Mathematically, the parallel tracks themselves may be viewed as corresponding to the parallel lines $y = 0$ and $y = 1$ in the coordinate plane, and the track going from one to the other may be viewed as a regular smooth curve that is the graph of a twice differentiable function f such that $f(x)$ is zero if $t \leq 0$, $f(x) = 1$ if $t \geq 1$, and on $[0, 1]$ the function f is given by a polynomial $p(x)$. The existence of a second derivative ensures that the slope of the tangent line would be a continuous function of x, and in addition we want to assume that *the curvature is also a continuous function of* x. Find a polynomial $p(x)$ of degree 5 such that all the required conditions are fulfilled. [*Hint:* If we are given a graph curve with parametric equatitons $(t,\ y(t))$, then the curvature at parameter value t is given by the formula

$$k(t) \quad = \quad \frac{|y''|}{\left(1 + (y')^2\right)^{3/2}}$$

and one step in the argument is to use this fact to compute $p''(0)$ and $p''(1)$. In fact, the conditions of the problem uniquely specify the values of p and its first and second derivatives at both 0 and 1. Why does this mean the only values to find are the coefficients of x^3, x^4 and x^5?]

Optional. Graph the function f using calculator or computer graphics.

I.5 : Frenet-Serret Formulas

(O'Neill, §§ 2.3–2.4)

O'Neill, pp. 64–66: 1, 5

Additional exercises

1. Let $\mathbf{x}$ be a regular smooth curve with a continuous third derivative, and let $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ be its Frenet trihedron. Prove that there is a vector $\mathbf{W}$ (the *Darboux vector*) such that $\mathbf{T}' = \mathbf{W} \times \mathbf{T}$, $\mathbf{N}' = \mathbf{W} \times \mathbf{N}$, and $\mathbf{B}' = \mathbf{W} \times \mathbf{B}$. What is the length of $\mathbf{W}$?

2. If $\mathbf{x}$ is defined for $t > 0$ by the formula

$$\mathbf{x}(\mathbf{t}) \quad = \quad \left(t, \ \frac{1+t}{t}, \ \frac{1-t^2}{t} \right)$$

show that $\mathbf{x}$ is planar.

II. Topics from Multivariable Calculus and Geometry

II.1 : Differential forms

(O'Neill, §§ 1.5–1.6)

O'Neill, pp. 25–26: 5, 6 (first part only), 9 (last sentence only)
O'Neill, pp. 31–32: 1, 3–5

Additional exercise

1. Suppose that ω ia a 2-form on $\mathbf{R}^3$ such that $\omega \wedge dx = 0$. Explain why there is a 1-form θ such that $\omega = \theta \wedge dx$.

II.2 : Smooth mappings

(O'Neill, §§ 1.7, 3.2)

Additional exercises

Definition. A subset K of $\mathbf{R}^n$ is said to be *convex* if whenever $\mathbf{x}$ and $\mathbf{y}$ lie in K then the whole line segment defined by the parametrized curve $\mathbf{x} + t\,(\mathbf{y} - \mathbf{x})$ for $t \in [0,1]$ is contained in K.

1. Prove that an open convex set is a connected domain [*Hint:* Imitate the proof for the set of all point whose distance from some point $\mathbf{p}$ is less than some positive number r.].

2. Show by example that an intersection of two connected domains in $\mathbf{R}^2$ is not necessarily a connected domain. [*Hint:* Let U be the annular region defined by the inequalities $1 < x^2 + y^2 < 9$

and let V be the horizontal strip defined by the inequality $|y| < \frac{1}{2}$. Verify that U is arcwise connected using the polar coordinate mapping, which yields a continuous 1–1 mapping from the convex set $(1,3) \times [0, 2\pi)$ onto U. If $U \cap V$ were connected then by a result in the Appendix to Chapter 5 in do Carmo, it would also be arcwise connected. Suppose now that $\mathbf{x}$ is a curve joining the points (± 2.0). By the Intermediate Value Theorem there must be some parameter value t_0 such that the first coordinate of $\mathbf{x}(t_0)$ is equal to zero. Why does this mean that $\mathbf{x}$ cannot lie entirely inside $U \cap V$?]

3. Given an matrix A with real entries , let $|A|$ denote the Euclidean length given by the square root of the standard sum $\sum_{i,j} |a_{i,j}|^2$. If P and Q are two matrices with real entries such that the product PQ can be defined, prove that $|PQ| \leq |P| \cdot |Q|$.

4. Let U be a convex connected domain in $\mathbf{R}^n$, and let $f : U \to \mathbf{R}^m$ be a smooth $\mathcal{C}^1$ function.

(a) Prove that

$$f(\mathbf{y}) - f(\mathbf{x}) \quad = \quad \int_0^1 \left([Df(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))](\mathbf{y} - \mathbf{x}) \right) \, dt$$

for all $\mathbf{x}$, $\mathbf{y} \in U$. [*Hint:* Explain why the integrand is the derivative of the function

$$f(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))$$

using the Chain Rule.]

(b) Suppose that the derivative matrix function Df satisfies $|Df| \leq M$ on U. Prove that

$$|f(\mathbf{y}) - f(\mathbf{x})| \quad \leq \quad M \cdot |\mathbf{y} - \mathbf{x}|$$

for all $\mathbf{x}$, $\mathbf{y} \in U$.

Note. An inequality of this sort is called a *Lipschitz condition.*

II.3 : Inverse and Implicit Function Theorems

(O'Neill, § 1.7)

Additional exercises

1. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a $\mathcal{C}^r$ function such that its derivative f' is everywhere positive and the limits of $f(t)$ as $t \to \pm\infty$ are $\pm\infty$ respectively. Prove that f has a $\mathcal{C}^r$ inverse function.

2. Prove that $F(x,y) = (e^x + y,\ x - y)$ defines a 1–1 onto $\mathcal{C}^\infty$ map from $\mathbf{R}^2$ to itself with a $\mathcal{C}^\infty$ inverse.

3. Prove that $F(x,y) = (xe^y + y,\ xe^y - y)$ defines a 1–1 onto $\mathcal{C}^\infty$ map from $\mathbf{R}^2$ to itself with a $\mathcal{C}^\infty$ inverse.

4. (a) Using the change of variables formula, explain briefly why the area of a set in $\mathbf{R}^2$ is the same as the area of its image under a rigid motion of the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where A is a rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(b) More generally, if we are given an arbitrary **affine** transformation as above, where the only condition on A is invertibility, how is the area of a set $\mathcal{F}$ related to the area of its image $T(\mathcal{F})$?

5. A smooth $\mathcal{C}^r$ mapping f from a connected domain $U \subset \mathbf{R}^2$ into $\mathbf{R}^2$ is said to be *regularly conformal* at $\mathbf{p} = (u_0, v_0) \in U$ if the Jacobian of f is positive and for all regular smooth curve pairs $\mathbf{x}$ and $\mathbf{y}$ satisfying $\mathbf{x}(s_0) = \mathbf{y}(s_0) = \mathbf{p}$ the angle between $\mathbf{x}'(s_0)$ and $\mathbf{y}'(s_0)$ is equal to the angle between $[f \circ \mathbf{x}]'(s_0)$ and $[f \circ \mathbf{y}]'(s_0)$.

(a) Prove that the partial derivatives of the coordinate functions satisfy the *Cauchy-Riemann equations*:

$$\frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2}\,, \qquad \frac{\partial f_2}{\partial x_1} = -\frac{\partial f_1}{\partial x_2}$$

[*Hint:* If $A = Df(\mathbf{p})$, one needs to show that $\cos \angle(A\,\mathbf{x},\ A\,\mathbf{y}) = \cos \angle(\mathbf{x},\ \mathbf{y})$ for all nonzero vectore $\mathbf{x}$ and $\mathbf{y}$. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ denote the columns of A, and let J denote counterclockwise rotation through $\pi/2$. Why is $\mathbf{a}_2 = c\,J(\mathbf{a}_1)$ for some constant c, and why does the determinant condition imply c is positive? Explain why $A(\mathbf{e}_1 + \mathbf{e}_2) = \mathbf{a}_1 + \mathbf{a}_2$ must be perpendicular to $A(\mathbf{e}_1 - \mathbf{e}_2) = \mathbf{a}_1 - \mathbf{a}_2$, and use this to conclude that $c = 1$.]

(b) There is a modified version of this relation that holds among the partial derivatives if the Jacobian is **negative**. State it and explain why it is true. [*Hint:* Consider what happens if one composes f with the reflection map $S(x, y) = (x,\ -y)$.]

Note. Functions satisfying the Cauchy-Riemann equations are also known as *complex analytic* functions, and they are the central objects studied in complex variables courses.

II.4 : Congruence of geometric figures

(O'Neill, §§ 3.1, 3.4–3.5)

do Carmo, § 1–7, pp. 47-50: 1, 3, 15

Additional exercises

1. Let F be an isometry of $\mathbf{R}^n$, and let $\mathbf{x}$ and $\mathbf{y}$ be distinct points of $\mathbf{R}^n$ such that $F(\mathbf{x}) = \mathbf{x}$ and $F(\mathbf{y}) = \mathbf{y}$. Suppose that $\mathbf{z}$ is a point on the line joining $\mathbf{x}$ to $\mathbf{y}$ that can be expressed as $\mathbf{z} = t\mathbf{x} + (1-t)\mathbf{y}$ for some scalar t. Prove that $F(\mathbf{z}) = \mathbf{z}$ also holds. [*Hints:* Use the fact that $F(\mathbf{w}) = A(\mathbf{w}) + \mathbf{b}$ for some linear transformation A along with the identity $\mathbf{b} = t\mathbf{b} + (1-t)\mathbf{b}$.]

2. Prove that congruent curves have equal lengths.