Wedge products and vector operations

The wedge product also has some important relationships to cross products and triple products of 3-dimensional vectors. Specifically, the following conclusions should be added to the theorem starting on page 42 of the class notes and continuing onto page 43. The items are numbered to follow the conclusions already stated in the theorem.

(iv) *Suppose that* $p = 2$ *and* $n = 3$*, and let* $\mathbf{F}$, $\mathbf{G}$ *and* $\mathbf{H}$ *be 3-dimensional vector fields. In the preceding notation we have* $\omega_{\mathbf{F}} \wedge \omega_{\mathbf{G}} = \Omega_{\mathbf{F}\times\mathbf{G}}$.

(v) *In the notation of the preceding item, we also have*

$$\omega_{\mathbf{F}} \wedge \omega_{\mathbf{G}} \wedge \omega_{\mathbf{H}} \quad = \quad [\mathbf{F}, \mathbf{G}, \mathbf{H}] \cdot dx \wedge dy \wedge dz \ .$$

The following exercise establishes some further results:

Let f be a real valued function on $\mathbb{R}^3$ with continuous second partial derivatives, and let $\mathbf{G}$ be a 3-dimensional vector field on $\mathbb{R}^3$. Let ω be the differential 1-form corresponding to $\mathbf{G}$.

(a) Express the differential 2-form θ corresponding to the vector field $\nabla f \times \mathbf{G}$ in terms of f and ω.

(b) Given a differential 1-form $\omega = P\,dx + Q\,dy + R\,dz$ and a differential 2-form

$$\theta = L\,dy \wedge dz + M\,dz \wedge dx + R\,dx \wedge dy$$

the wedge product $\omega \wedge \theta$ is a differential 3-form which may be written as $h\,dx \wedge dy \wedge dz$ for some real valued function h. Express h in terms of the functions L, M, N, P, Q, R.