The Schönflies Theorem in the Euclidean plane

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The purpose of this document is to prove the following result that was stated in `dgnotes.pdf`:

JORDAN-SCHÖNFLIES THEOREM *Let* $\mathbf{x}$ *be a regular smooth simple closed curve in the plane of class* C^r *where* r *is* SUFFICIENTLY LARGE, *and let* Γ *be the image of* $\mathbf{x}$. *Assume for the sake of definiteness that the domain of definition for* $\mathbf{x}$ *is the interval* $[0,\, 2\pi]$. *Then the following conclusions hold:*

(i) *The complementary set* $\mathbf{R}^2 - \Gamma$ *consists of two connected domains, exactly one of which is bounded. Furthermore, every point of* Γ *is a boundary point for each of these regions.*

(ii) *There is a small positive number* ε *for which there is a* $1-1$ C^r *map* $\mathbf{Y}$ *from the open disk* $N_{1+\varepsilon}(\mathbf{0})$ *in the plane into* $\mathbf{R}^2$ *such that* $D\,\mathbf{Y}$ *is always invertible, the map* $\mathbf{Y}$ *sends* $N_1(\mathbf{0})$ *to the bounded connected domain in* $\mathbf{R}^2 - \Gamma$, *and the map* $\mathbf{Y}$ *extends* $\mathbf{x}$ *in the sense that* $\mathbf{Y}(\cos t, \sin t) = \mathbf{x}(t)$ *for all* $t \in [0,\, 2\pi]$.

(iii) *In the setting above, suppose that the bounded connected domain* $\mathbf{Inside}(\Gamma)$ *in* $\mathbf{R}^2 - \Gamma$ *contains* $\mathbf{0}$. *Then one can choose* $\mathbf{Y}$ *so that it is either the identity map or the reflection* $S(u,v) = (u,-v)$ *on some disk of radius* δ, *where* $\delta > 0$ *is chosen so that* $N_{2\delta}(\mathbf{0}) \subset \mathbf{Inside}(\Gamma)$.

The proof of this result requires a considerable amount of background beyond the prerequisites for an introductory course in differential geometry. We shall describe a proof that relies on some basic facts from a standard first graduate course in algebraic topology as well as the theory of Morse functions. The book, *Topology of surfaces*, by A. Gramain (Edited English translation by L. Boron, C. Christenson and B. Smyth, BCS Associates, Moscow ID, 1984, ISBN 0-914351-01-X), discusses the topology of surfaces from a similar perspective at the undergraduate level.

References

Here are some additional references that we shall use in this document.

The following is a standard reference for a first course on point set topology, the fundamental group, and covering spaces:

All the algebraic topology we need is contained in the following online reference:

`http://www.math.cornell.edu/ hatcher/AT/ATpage.html`

Hard copies of this book are also available.

Finally, here are two classical but still outstanding references for Morse Theory:

J. W. Milnor. Lectures on the h-cobordism theorem (Notes by L. Siebenmann and J. Sondow), Princeton Mathematical Notes No. 1. Princeton University Press, Princeton, N. J., 1965. ISBN: 0-691-07996-X.

J. W. Milnor. Morse theory (Based on lecture notes by M. Spivak and R. Wells), Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N. J., 1963. ISBN: 0-691-08008-9.

The Jordan Curve Theorem

A fairly extensive discussion of this result is given in Section 61 of *Topology* (Second Edition) , by J. Munkres (Prentice-Hall, Upper Saddle River NJ, 2000, ISBN 0-13-181629-2).

JORDAN CURVE THEOREM *Let* $\mathbf{x}$ *be a continuous simple closed curve in the plane, and let* Γ *be the image of* $\mathbf{x}$*. Then the complementary set* $\mathbf{R}^2 - \Gamma$ *consists of two connected domains, exactly one of which is bounded. Furthermore, every point of* Γ *is a boundary point for each of these regions.*

A proof of this result is given on pages 377–381 of Munkres' book. As noted on page 376 of Munkres, this proof is longer than some arguments but it is relatively elementary.∎

Bicollar neighborhoods

The Jordan Curve Theorem is the first step in proving the Jordan-Schn̈flies Theorem as formulated above. One important consequence of the latter is that a regular smooth simple closed curve Γ has an open subset that looks like a standard annulus neighborhood of the unit circle defined by inequalities of the form

$$1 - \varepsilon \quad < \quad |z| \quad < \quad 1 + \varepsilon$$

and the next step in the proof is to prove a similar result for an arbitrary curve Γ satisfying the given conditions.

BICOLLAR NEIGHBORHOOD THEOREM. *Let* $\mathbf{x}$ *be a regular smooth simple closed curve in the plane of class* C^r *where* r *is* SUFFICIENTLY LARGE*, and let* Γ *be the image of* $\mathbf{x}$*. Assume for the sake of definiteness that the domain of definition for* $\mathbf{x}$ *is the interval* $[0,\, 2\pi]$*. Then there is a small positive number* ε *for which there is a* $1-1$ C^r *map* $\mathbf{Y}$ *from the annulus in the plane defined by the inequalities*

$$1 - \varepsilon \quad < \quad |z| \quad < \quad 1 + \varepsilon$$

into $\mathbf{R}^2$ *such that* $D\,\mathbf{Y}$ *is always invertible, the map* $\mathbf{Y}$ *sends* $N_1(\mathbf{0})$ *to the bounded connected domain in* $\mathbf{R}^2 - \Gamma$*, and the map* $\mathbf{Y}$ *extends* $\mathbf{x}$ *in the sense that* $\mathbf{Y}(\cos t, \sin t) = \mathbf{x}(t)$ *for all* $t \in [0,\, 2\pi]$*.*

A connected domain satisfying the conditions in the conclusion is called a *bicollar neighborhood* of the curve.

Proof. Let J be the orthogonal linear transformation on $\mathbf{R}^2$ defined by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(hence J represents clockwise rotation by 90 degrees). One can then extend $\mathbf{x}$ smoothly to a map

$$\mathbf{Y}_0 : \mathbf{R}^2 - \{\mathbf{0}\} \to \mathbf{R}^2$$

using polar coordinates as follows:

$$\mathbf{Y}_0(r,\, \theta) \quad = \quad \mathbf{x}(\theta) \ + \ \ln t \cdot J(\mathbf{x}'(\theta)\,)\ .$$

The Jacobian of this function at $(1,\, \theta)$ is then equal to $|\mathbf{x}'(\theta)|^2$ and consequently is positive at all such points. By the Inverse Function Theorem, there is some $\delta(\theta) > 0$ such that the restriction of

$\mathbf{Y}_0$ to all points in the open disk of radius $\delta(\theta)$ centered at $(1, \theta)$ is $1-1$, its image is a connected domain, and the restriction has a $\mathcal{C}^r$ inverse map g.

To complete the proof we need to show that there is an $\varepsilon > 0$ such that $\mathbf{Y}_0$ has a nonzero Jacobian for $r \in (1-\varepsilon,\, 1+\varepsilon)$ and $\mathbf{Y}_0$ is also 1–1 on this annular region. If so, then we may take $\mathbf{Y}$ to be the restriction of $\mathbf{Y}_0$ to this annulus. The verifications of both facts are essentially exercises in point set topology. For the first one, the crucial point is that there is a connected domain containing the unit circle on which the Jacobian is nonzero, and the proof is essentially given by Lemma 26.8 on pages 168–169 of Munkres. This result shows that there is some $\varepsilon_1 > 0$ such that the Jacobian is nonzero for $r \in (1-\varepsilon_1,\, 1+\varepsilon_1)$. On this set the map $\mathbf{Y}_0$ is locally 1–1; *i.e.*, for each point $\mathbf{a}$ on the annulus there is a small disk centered at $\mathbf{a}$ such that the restriction of $\mathbf{Y}_0$ to this small disk is 1–1. Under these conditions one can show that there is a possibly smaller connected domain containing the unit circle on which the restricted map is globally 1–1. This exercise and its solution appear on pages 4–5 of the following online document:

`http://www.math.ucr.edu/~res/math205A/solutions5.pdf`

One can now use the result from Munkres one more time to find the desired positive number ε.■

Note. One can use the extension $\mathbf{Y}_0$ to determine whether the sense of the simple closed curve is clockwise or counterclockwise as follows: Suppose that $\mathbf{0}$ lies in the inside region determined by the curve, and pick a parameter value t^* such that $|\mathbf{x}(t^*)|$ is maximized. Then standard differentiation formulas imply that the tangent vector $\mathbf{x}'(t^*)$ is perpendicular to $\mathbf{x}(t^*)$, and therefore the tangent vector is either a positive multiple of the tangent vector for a clockwise or counterclockwise circle through $\mathbf{x}(t^*)$ whose center is $\mathbf{0}$ and whose radius is $|\mathbf{x}(t^*)|$. The sense of the curve is the same as the sense of the corresponding circle whose tangent vector is a positive multiple of $\mathbf{x}'(t^*)$.

The Riemann Mapping Theorem

One corollary of the Jordan-Schönflies Theorem is that the "inside" region determined by a regular smooth simple closed curve in the plane a connected domain, and in fact we claim that the complement of this region also has the property that any two points can be joined by a continous curve that lies entirely in the complement.

PROPOSITION. *Let Γ be a regular smooth simple closed curve in the plane of class $\mathcal{C}^r$ where r is* SUFFICIENTLY LARGE*, let U and V be the two disjoint, connected domains that form the complement of $\mathbf{R}^2 - \Gamma$, and assume that U is the connected domain that is bounded. Then every pair of points in C can be joined by a regular piecewise smooth curve that lies entirely in C.*

Proof. Let C denote the complement of the inside region associated to Γ, and define a binary relation $\sim$ on C by $\mathbf{p} \sim \mathbf{q}$ if and only if there is a smooth curve $\mathbf{x}$ joining $\mathbf{p}$ to $\mathbf{q}$. Reparametrizing if necessary, we may as well assume that the left hand point of the interval of definition for $\mathbf{x}$ is equal to 0. We claim this is an equivalence relation on C. The curve $\mathbf{x}(t) = \mathbf{p}$ for all $t \in [0,1]$ is a curve joining $\mathbf{p}$ to itself, so we have $\mathbf{p} \sim \mathbf{p}$. If $\mathbf{p} \sim \mathbf{q}$ then there is a continuous curve $\mathbf{x}$ whose image lies in C such that $\mathbf{x}(0) = \mathbf{p}$ and $\mathbf{x}(L) = \mathbf{q}$ for some $L > 0$. If $\mathbf{x}^{\mathrm{OP}}$ denotes the curve defined by $\mathbf{x}^{\mathrm{OP}}(t) = \mathbf{x}(L-t)$, then $\mathbf{x}^{\mathrm{OP}}(0) = \mathbf{q}$ and $\mathbf{x}^{\mathrm{OP}}(L) = \mathbf{p}$ for some $L > 0$, so that $\mathbf{p} \sim \mathbf{q}$. Finally, if $\mathbf{p} \sim \mathbf{q}$ and $\mathbf{q} \sim \mathbf{r}$ then there are continuous curvex $\mathbf{x}$ and $\mathbf{y}$ such that their images lie entirely in C with $\mathbf{x}(0) = \mathbf{p}$ and $\mathbf{x}(L) = \mathbf{q}$ and similarly $\mathbf{y}(0) = \mathbf{p}$ and $\mathbf{y}(M) = \mathbf{r}$ for some $L,\, M > 0$. Combine these curves to obtain a curve $\mathbf{z}$ defined on $[0, L+M]$ by the formula $\mathbf{z}(t) = \mathbf{x}(t)$ if $t \in [0, L]$ and $\mathbf{z}(t) = \mathbf{y}(t-L)$ if $t \in [L, L+M]$. Since these definitions agree at L, the resulting curve is continuous. By construction we have $\mathbf{z}(0) = \mathbf{p}$ and $\mathbf{x}(L+M) = \mathbf{r}$. Therefore we have an equivalence relation that partitions C into pairwise disjoint equivalence classes. We claim there is precisely one equivalence class; if so this proves the desired conclusion.

The set C is a union $\Gamma \cup V$ where Γ is the original curve and V is the "outside" region. By the proof of the Bicollaring Theorem we know that for each point $\mathbf{a}$ on the curve there is a smooth 1-1 map $\mathbf{g}$ from some open disk D centered at $\mathbf{0}$ into $\mathbf{R}^2$ such that $\mathbf{g}(\mathbf{0}) = \mathbf{a}$, the image of $\mathbf{g}$ is a connected domain containing $\mathbf{a}$, and $\mathbf{g}(u, v)$ lies on Γ if and only if $v = 0$.

Consider the sets of all points $W_{\pm}$ in $\mathbf{R}^2$ that have the form $\mathbf{g}(u, v)$ where $v > 0$ for W_+ and $v < 0$ for W_-. Each of these sets is a connected domain that lies in the complement of Γ, and it follows that each is contained in either the inside or outside region associated to Γ. We claim that one of $W_{\pm}$ is contained in the inside region and the other is contained in the outside region. For each of the sets $W_{\pm}$ we know that given two points in the set then there is a regular piecewise smooth curve joining them; if, say, one of these points lies in the inside region determined by Γ and another point lies in the outside region, then by the Jordan Curve Theorem the curve joining these points would have to contain a point of Γ, and we know this does not happen. This implies that each of $W_{\pm}$ is either contained in the inside or outside region; we need to explain why both $W_{\pm}$ cannot belong to one of these two regions. If this did happen, then there would be a small open disk centered about $\mathbf{a}$ that did not contain any points from the second of the regions determined by the curve. But we know that every point on Γ is simultaneously a limit of two sequences of points, one lying in the inside region and the other lying in the outside region (in fact, every point on Γ is a limit of infinitely many sequences of each type). Therefore we cannot have a situation where both $W_{\pm}$ lie either in the inside or the outside region determined by Γ.

There are now two cases depending upon whether or not W_- lies in the inside region determined by Γ. Replacing $\mathbf{g}(u, v)$ by $bfg(u, -v)$ if necessary, we may assume that W_- does line in the inside region and hence W_+ lies in the outside region. Consider the curve $\beta(t) = \mathbf{g}(0, t)$ for $t \geq 0$ sufficiently close to 0. This is a smooth curve that lies entirely in C with one endpoint equal to $\mathbf{a}$ and the other endpoint belonging to the outside region V. Therefore it follows that the point $\mathbf{a} \in \Gamma$ is equivalent under $\sim$ to a point in V. Since the points of V all lie in one equivalence class and $\mathbf{a}$ was arbitrary, it follows that $\sim$ has only one equivalence class.■

The preceding result implies that the inside region U is simply connected in the sense of Ahlfors' text on complex analysis (see the reference below). This formulation of simple connectivity is radically different from the one given in Munkres' text (see page 333), but for connected domains in $\mathbf{R}^2$ they are equivalent by the Riemann Mapping Theorem which is discussed below.

Note. For more general spaces the definition in Munkres is the accepted one, and the condition in Ahlfors does **NOT** imply the condition in Munkres for connected domains in $\mathbf{R}^3$ and vice versa. A basic counterexample to show that the Ahlfors condition does not imply the Munkres condition in $\mathbf{R}^3$ is given by the connected "solid torus" domain obtained by rotating the interior of the circle defined by $(x-2)^2 + y^2 = 1$ about the y-axis, and a basic example in $\mathbf{R}^3$ for the other direction in given by taking the annulus defined by the inequalities $1 \leq |\mathbf{v}| \leq 2$.

The RIEMANN MAPPING THEOREM states that every simply connected domain U in the plane $\mathbf{R}^2$ is either equal to $\mathbf{R}^2$ itself or else there is a 1–1 onto smooth map f from U to the open unit disk D such that the Jacobian of f is nonzero at each point and if $f(x, y) = (\, u(s, y),\ v(x, y)\,)$, then u and v satisfy the Cauchy Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

This means that the "inside" connected domain associated to a simple closed curve in the plane can always be mapped in a smooth 1–1 onto manner to the inside domain associated to the standard unit circle. A proof of this result appears on pages 229–235 of *Complex Analysis* (Third Edition)

, by L. Ahlfors (International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1979, ISBN 0-07-000657-1).

Final remark. If Γ, U and V are as in the proof of the proposition above, a similar argument shows that $\mathrm{B} = U \cap \Gamma$ also satisfies the property in the conclusion: *Every pair of points in B can be joined by a regular piecewise smooth curve that lies entirely in B.*■

A recognition principle

The next step gives a sufficient condition for the Jordan-Schönflies Theorem to hold; the remaining steps in the proof are essentially aimed at showing the sufficient condition holds in all cases.

WEAK JORDAN-SCHÓNFLIES THEOREM. *Let Γ be a regular smooth simple closed curve in the plane of class C^r where r is* SUFFICIENTLY LARGE, *let U and V be the two disjoint, connected domains that form the complement of $\mathbf{R}^2 - \Gamma$, and assume that U is the connected domain that is bounded. Suppose that there is a smooth function h defined on a connected domain W containing U and Γ such that h is identically 1 on Γ while h takes values in $[0,1)$ and $(,+\infty)$ on U and V respectively. Assume further that $h(\mathbf{v}) = |\mathbf{v}|^2$ near $\mathbf{0}$ and that $\mathbf{0}$ is the only point of W at which $\nabla h = \mathbf{0}$.. Then the following conclusions hold:*

(*i*) *There is a small positive number ε for which there is a $1-1$ C^r map $\mathbf{Y}$ from the open disk $N_{1+\varepsilon}(\mathbf{0})$ in the plane into $\mathbf{R}^2$ such that $D\mathbf{Y}$ is always invertible, the map $\mathbf{Y}$ sends $N_1(\mathbf{0})$ to the bounded connected domain in $\mathbf{R}^2 - \Gamma$, and the map $\mathbf{Y}$ extends $\mathbf{x}$ in the sense that $\mathbf{Y}(\cos t, \sin t) = \mathbf{x}(t)$ for all $t \in [0,\ 2\pi]$.*

(*ii*) *One can choose $\mathbf{Y}$ so that it is either the identity map or the reflection $S(u,v) = (u,-v)$ on some disk of radius δ, where $\delta > 0$ is chosen so that $N_{2\delta}(\mathbf{0}) \subset \mathbf{Inside}(\Gamma)$.*

The arguments proving this this result are given on pages 12–13 of Section I.3 in *Morse Theory*, pages 23–23 of Section 3 in *Lectures on the h-cobordism Theorem* and also discussed in Proposition 1 on page 56 of Gramain's book.■

Nondegenerate critical points

The next objective is to construct a smooth function on the set B with as few critical points as possible, with all such points relatively far away from the boundary curve Γ.

EXISTENCE THEOREM. *There is a smooth function h defined on some connected domain W containing the curve Γ such that the restriction of the function to B takes values in some interval $[0,a]$ for some $a > 0$, the function's value on Γ is always equal to a, its value on U is always less than a, it has no critical points close to Γ, and at each critical point $\mathbf{p}$ the Hessian matrix with (i,j) entry equal to*

$$\frac{\partial^2 h}{\partial x_i\, \partial x_j}\ (\mathbf{p})$$

has nonzero determinant.

A critical point $\mathbf{p}$ is said to be *nondegenerate* if the determinant of the Hessian is nonzero. It follows from a Taylor series approximation (see pages 6–8 of Section I.2 in *Morse Theory*) that nondegenerate critical points are isolated from each other and there are only finitely many of them in the inside region determined by Γ.

Note that any function h as above must have at least one critical point because the function has nonzero gradients at points of Γ, which implies that the function is not constant at points of Γ. Since the values of the function lie in $[0, a]$, this implies that the function must take some values strictly less than a, and therefore there must be some point of U where the function takes a minimum value. At this point the gradient of h must be equal to zero.

COMPLEMENT TO THE EXISTENCE THEOREM. *If* $\mathbf{0}$ *belongs to* U*, then one can choose* h *so that* $h(\mathbf{v}) = |\,\mathbf{v}\,|^2$ *for* $\mathbf{v}$ *sufficiently close to* $\mathbf{0}$.

Proofs of these results are given on pages 32–38 of Section I.6 in *Morse Theory* and pages 7–19 in Section 2 of *Lectures on the h-cobordism Theorem.*■

Definition. Functions of the type described in the Existence Theorem are called *Morse functions*. They play an important role in many different areas of topology and geometry as well as their applications to other subjects. Functions with all the properties in the Complement will be called based Morse functions.

Cancellation of critical points

In view of the results in the preceding two sections, the proof of the Jordan Schónflies Theorem reduces to showing that one can find a Morse function that only has one critical point; namely, the origin. The latter is an absolute minimum, and all remaining critical points are either relative minima, relative maxima, or saddle points. In the language of Morse Theory, these are described as nondegenerate critical points of index 0, 2 and 1 respectively.

At this point it is necessary (or at least extremely convenient) to use some algebraic topology. Given a topological space X and a subspace $A \subset X$ one can define a sequence of so-called *singular homology groups* $H_n(X, A)$, where n runs over all positive integers; a full account of the motivation and construction appears in Hatcher's online book. For our purposes it will suffice to consider the singular homology with rational coefficients, in which case the groups are vector spaces over the rational numbers. If A is the empty set then one denotes this homology group by $H_n(X)$. The following result summarizes all the specific information that we shall need:

THEOREM. *Let* Γ *be a simple closed curve in* $\mathbf{R}^2$*, let* U *be the inside region associated to* Γ*, and let* B *denote the union* $U \cap \Gamma$*. Suppose that* D *is an open disk of radius* ε *centered at the origin that is contained in* B*, construct* B_0 *from* B *by removing the correspnding open disk of radius* $\varepsilon/2$*, and let* Γ_0 *be the circle of radius* $\varepsilon/2$ *centered at* $\mathbf{0}$*. Then* $H_0(B_0, \Gamma)$*,* $H_1(B_0, Gamma)$ *and* $H_0(B_0, \Gamma_0)$ *are all zero-dimensional rational vector spaces.*

Comments and references. The statement $\dim H_0(X, A) = 0$ is equivalent to saying that for every point x in X there is a continuous curve joining x to some point in A, and the condition $\dim H_1(X, A) = 0$ is true if every two points of A can be joined by a continuous curve and X is simply connected. Here are some specific references for the assertions regarding the homology groups:

[1] *The vanishing of* $H_0(X, A)$ *is equivalent to the statement that every point in* X *can be joined to a point in* A *bv a continuous curve.* The follows by combining the discussions on pages 108–110 and 115–117 of Hatcher's book.

[2] *Simply connected spaces have trivial* 1*-dimensional homology.* This is discussed in Section 2A on pages 166–168 of Hatcher's book.

We have already noted that every Morse function on B must have at least one critical point which is an absolute minimum and thus has index zero. We want to find a based Morse function on B that has this minimum number of critical points. In particular, we would like to minimize the numbers of relative minima and maxima. The following result provides the means for doing so:

ELIMINATION OF NONZERO RELATIVE EXTREMA. *Let h be a based Morse function on B as above. Then there is another Morse function h_1 on B such that h_1 and h are equal near $\mathbf{0}$ and Γ, but h_1 has no other relative extrema away from Γ. Furthermore, the number of saddle points for h_1 is less than or equal to the number of saddle points for h.*

Comments on the proof. By the first part of Theorem 8.1 on pages 100–101 of *Lectures on the h-cobordism Theorem*, the conditions on zero-dimensional homology imply that one can construct a new Morse function h_0 that agrees with the old one on the specified points such that all critical points of index 0 are eliminated, a similar number of critical points of index 1 are eliminated, and no additional critical points are created. One then considers the negative function $-h_0$. The critical points of h_0 and $-h_0$ are the same but the index of a critical point changes; specifically, critical points of index λ for h_0 become critical points of index $2 - \lambda$ for $-h_0$. By the previous step we have eliminated critical points of index 2 except for $\mathbf{0}$, and the remaining critical points for $-h_0$ have index 0 or 1. One can then apply Theorem 8.1 to construct a new Morse function h_2 that agrees with $-h_0$ on the specified sets such that all critical points of index 0 are eliminated and no new critical points are created. It follows that the only relative extremum of h_2 is the absolute maximum at $\mathbf{0}$. If we take $h_1 = -h_2$ then we have a function with the properties described in the conclusion of the result.∎

Essentially the same argument is employed on pages 70–76 of Gramain's book; the argument given there is simpler than the one in Milnor's book because it concentrates on the 2-dimensional case, but the result in Gramain's book is stated in somewhat different terms. In both cases the essential conclusion is that nonzero critical points of index 0 and 2 can be eliminated (Milnor) or ignored (Gramain).

The number of saddle points

We now have a based Morse function h on B such that $\mathbf{0}$ is an absolute minimum and there are no other relative extrema. Therefore all nonzero critical points of h on B have index 1; *i.e.*, they are all saddle points. Let N_k be the number of nonzero critical points of h with index k. Then we have $N_2 = 0 = N_0$, and N_1 is some nonnegative integer to be determined. If we can show that $N_1 = 0$, then we can apply the previous results to deduce the Jordan-Schönflies Theorem in its full strength.

We have seen that homology groups sometimes provide a means for analyzing the numbers of critical points for a Morse function, and in fact the result discussed above is just the most basic case of some far-reaching results. Milnor's books describe several of these results. For our purposes it will suffice to use the small portion of the latter given by the *Morse inequalities*. These are discussed on pages 28–31 in Section I.5 of *Morse Theory.*

Specifically, the information we need is contained in the last paragraph on page 30. According to the formula in the fourth line from the bottom of the page, since $N_2 = 0$ we have

$$\dim H_1(B_0, \Gamma_0) \;-\; \dim H_0(B_0, \Gamma_0) \;=\; N_1 \;-\; N_0 \;=\; N_1 \;.$$

However, we know that the dimensions of both homology groups on the left hand side are equal to zero, and therefore we must also have $N_1 = 0$.

The results of this section and the preceding one yield the following strong conclusion:

THEOREM. *If B is given as above, then B admits a based Morse function such that there are no critical points aside from the absolute minimum at* $\mathbf{0}$.∎

If we combine this result with the Weak Jordan-Schönflies Theorem stated previously, we obtain the full strength version of the theorem as stated at the beginning of this document.∎