Math 144
Fall 2019

SOLUTIONS TO THE MIDTERM PRACTICE PROBLEMS

1. (a) For the first identity, note that by definition

$$A + \emptyset \quad = \quad (A - \emptyset) \cup (\emptyset - A)$$

and the latter is equal to A because $A - \emptyset = A$ and $\emptyset - A = \emptyset$. The second one follows because

$$A + A \quad = \quad (A - A) \cup (A - A) \quad = \quad \emptyset \cup \emptyset$$

which is just the empty set.■

(b) The suggestion in the hint is to look at the 8 mutually exclusive possibities, indexed by the binary expansions of 0 through 7:

(000) $x \notin A$, $x \notin B$, $x \notin C$
(001) $x \notin A$, $x \notin B$, $x \in C$
(010) $x \notin A$, $x \in B$, $x \notin C$
(011) $x \notin A$, $x \in B$, $x \in C$
(100) $x \in A$, $x \notin B$, $x \notin C$
(101) $x \in A$, $x \notin B$, $x \in C$
(110) $x \in A$, $x \in B$, $x \notin C$
(111) $x \in A$, $x \in B$, $x \in C$

In the cases where there are either two or no zeros, one checks that x belongs to both $A + (B + C)$ and $(A + B) + C$, and in the cases where the number of zeros is 1 or 3, one checks that x belongs to neither of these sets. The Venn diagram in `aabNewUpdate05a.144.f17.pdf` may be helpful for seeing this.■

(c) Follow the hint: If $A + C = B + C$ then $(A + C) + C = (B + C) + C$. By ($b$) the preceding is equivalent to $A + (C + C) = B + (C + C)$. But now ($a$) implies that this is equivalent to $A + \emptyset = B + \emptyset$, and by ($a$) the latter is equivalent to $A = B$.■

2. We shall first verify that $A \subset \bigcup_{a \in A} \{a\}$. If $a \in A$, then by definition we have $\{a\} \subset A$; since this is true for all $a \in A$, it follows that A is contained in the union. Conversely, suppose that $x \in \bigcup_{a \in A} \{a\}$. Then $x \in \{a\}$ for some $a \in A$, and by definition this implies that $x = a \in A$.■

If $A = \{a_0\}$, then the intersection is just $\{a_0\}$. On the other hand, if A has more than one element, let x, y be distinct members of A. Then $\cap_{a \in A} \{a\}$ is contained in the intersection $\{x\} \cap \{y\}$, and this is empty because $x \neq y$.■

3.

(a) The relation is reflexive and symmetric but not transitive. To see that it is not transitive, consider the related pairs $(a, b) \sim (b, c)$ and $(b, c) \sim (c, d)$ where different letters represent different elementsof the set. By the latter condition we knowthat (a, b) and (c, d) are not related.■

(b) Denote the original relation by $\sim$ and the equivalence relation by $\equiv$. If (a, b) and (x, y) are distinct elements of $A \times A$ then we have $(a, b) \sim (x, b)$ and $(x, b) \sim (x, y)$. By hypothesis we also have $(a, b) \equiv (x, b)$ and $(x, b) \equiv (x, y)$, and by transitivity these imply that $(a, b) \equiv (x, y)$.■

4. The interval J is order-isomorphic to $[1,2]$ because the map $f(x) = x+1$ defines a 1–1, onto and (strictly) order-preserving map from $[0,1]$ to $[1,2]$. However, J is not order-isomorphic to any of the others. We can see this using Theorem IV.6.2 because J has a minimal element and a maximal element, but

(*a*) the interval $[0,1)$ has no maximal element,

(*b*) the interval $(0,1]$ has no minimal element,

(*c*) the interval $(0,1)$ has neither a maximal element nor a minimal element.■

5. (*i*) For $f(x) = 1/(x^2+1)$ the function is 1–1 on the intervals $(-\infty, 0]$ and $[0,\infty)$. For $f(x) = 1/(2+\sin x)$ the function is 1–1 on every interval of the form $\big[\,(k-\frac{1}{2})\pi, (k+\frac{1}{2})\pi\big]$. For $f(x) = 1/(e^x+3)$ the function is 1–1 on the entire real line, so it is 1–1 on every closed interval $[a,b]$.■

(*ii*) (*i*) For $f(x) = 1/(x^2+1)$ the image of the function is $(0,1]$, so its image contains every closed interval $[h,1]$ such that $0 < h < 1$. For $f(x) = 1/(2+\sin x)$ the image of the is $\big[\frac{1}{3}, 1\big]$, so its image contains this interval. For $f(x) = 1/(e^x+3)$ the image of the function is $\big(0, \frac{1}{3}\big]$, so its image contains every closed interval $\big[h, \frac{1}{3}\big]$ such that $0 < h < \frac{1}{3}$.■

6. If n is even, then every $y \in [0,1]$ is x^n for a unique value of x such that $x \in (0,1]$, $x = 0$ or $x \in (-1,0]$, and also $(-x)^n = x^n$. Furthermore, if $|x| > 1$ then $x^n > 1$. Therefore the inverse image is $[-1,1]$.

If n is odd, then every real number y is x^n for some unique value of n, and furthermore $x \in [0,1]$ if and only if $y \in [0,1]$. Therefore the inverse image in this case is $[0,1]$.■

7. (*a*) If $f(x) = x^2$, then f maps the interval $A = [1,2]$ onto the interval $[1,4]$. To find $f^{-1}[A]$, we need to find all x so that $1 \leq x^2 \leq 2$. The latter is equivalent to $1 \leq |x| \leq \sqrt{2}$, and therefore the inverse image is $[-\sqrt{2}, -1] \cup [1, \sqrt{2}]$.■

(*b*) If $f(x) = 3x+4$, then f maps the interval $A = [0,1]$ onto the interval $[4,7]$. To find the inverse image, we need to find all x such that $3x+4 \in [0,1]$. The function $3x+4$ is strictly increasing, so if we can solve $3a+4 = 0$ and $3b+4 = 1$, then the inverse image will be the interval $[a,b]$. Standard algebra implies that $a = -\frac{4}{3}$ and $b = -\frac{1}{3}$, so the inverse image is the interval $\big[-\frac{4}{3}, -\frac{1}{3}\big]$.■