Math 144
Fall 2019

UPDATED GENERAL INFORMATION — NOVEMBER 22, 2019

More exercises on cardinal numbers

0. What is the cardinality of the set $\mathbb{Q}^n$ where $n \geq 2$ is an integer? Prove that your answer is correct.

1. Prove the following facts about cardinal numbers α, β, γ:

(a) If $\alpha \cdot \beta = 0$ then either $\alpha = 0$ or $\beta = 0$. [*Hint:* This is equivalent to showig that if $\alpha \geq 1$ and $\beta \geq 1$, then $\alpha \cdot \beta \geq 1$.]

(b) If $\alpha > 0$ then $0^\alpha = 0$, and if $\alpha = 0$ then $0^\alpha = 1$.

2. Use Corollary VII.4.3 to prove the following case of Additional Exercise VII.4.1: If α, α', β and β' are infinite cardinal numbers then $\alpha + \beta < \alpha' \cdot \beta'$.

Note. The cited exercise is also known as the Zermelo-König Theorem.

RECALL that infinite sums and products of cardinal numbers are defined in Section VII.1 of `set-theory-notes.pdf`.

3. Prove an infinite distributive law: $\alpha \cdot \sum_j \beta_j = \sum_j \alpha \cdot \beta_j$.

4. Prove the following infinite associativity properties for an indexed family of cardinal numbers $\alpha_{i,j}$:

$$\sum_i \left(\sum_j \alpha_{i,j} \right) = \sum_{i,j} \alpha_{i,j} = \sum_j \left(\sum_i \alpha_{i,j} \right)$$

$$\prod_i \left(\prod_j \alpha_{i,j} \right) = \prod_{i,j} \alpha_{i,j} = \prod_j \left(\prod_i \alpha_{i,j} \right)$$

5. Determine whether the following statement is true or false. If it is true, give a proof; if not, give an counterexample:

If $\{\alpha_j\}$ is a family of cardinal numbers indexed by J and I is strictly contained in J, then

$$\sum_{i \in I} \alpha_i < \sum_{j \in J} \alpha_j .$$

6. Let $\{A_j\}$ be an indexed family of subsets of the set X, where j runs through all values in the indexing set J. Prove the following inequality of cardinal numbers:

$$\sum_j |A_\alpha| \leq |X| \cdot |J|$$