UPDATED GENERAL INFORMATION — NOVEMBER 25, 2019

Solutions to exercises on cardinal numbers

0. The cardinality is $\aleph_0$ because $|\mathbb{Q}| = \aleph_0$, so that $|\mathbb{Q}|^n = \aleph_0^n = \aleph_0$.■

1. (a) Use the hint. Suppose that $1 \leq \alpha = |A|$ and $1 \leq \beta = |B|$. Then there is some $a \in A$ and some $b \in B$, and therefore we have an ordered pair $(a,b) \in A \times B$. But this means that $0 < |A \times B| = \alpha \cdot \beta$.■

(b) The first statemenet holds because there are no functions which are defined in the nonempty set A and take values in the empty set. The second statement holds because there is an identity map from the empty set to itself and there are no other maps with this domain and codomain.■

2. By Corollary VII.4.3 we know that $\alpha + \beta$ is the larger of α and β. If β is the larger one, then we have

$$\alpha + \beta \quad = \quad \beta \quad < \quad \beta' \quad \leq \quad \alpha' \cdot \beta' \ .$$

If α is the larger cardinal number, then we can switch the roles of α and β, and of α' and β', to obtain a similar conclusion.■

3. Let $|A| = \alpha$ and $|B_j| = \beta_j$. Then the left hand side of the equation gives the cardinality of $A \times (\sqcup_j \, B_j)$, and the right hand side gives the cardinality of $\sqcup_j \, A \times B_j$. The identity follows because the map sending $(a, \, (b,j)\,)$ to $(\,(a,b), \, j)$ is 1–1 and onto.■

4. For each (i,j) let $A_{i,j}$ satisfy $|A_{i,j}| = \alpha_{i,j}$. Then the conclusion is a consequence of the following set-theoretic identities:

$$\bigcup_i \left(\bigcup_j \, A_{i,j} \times \{(i,j)\} \right) \quad = \quad \bigcup_{i,j} \, A_{i,j} \times \{(i,j)\} \quad = \quad \bigcup_j \left(\bigcup_i \, A_{i,j} \times \{(i,j)\} \right)$$

$$\prod_i \left(\prod_j \, A_{i,j} \right) \quad = \quad \prod_{i,j} \, A_{i,j} \quad = \quad \prod_j \left(\prod_i \, A_{i,j} \right) \quad ■$$

5. One of the simplest counterexamples is given by $\aleph_0 = \aleph_0 + \aleph_0$.■

6. By definition $\coprod_j \, A_\alpha = \cup_j \, A_\alpha \times \{j\} \subset X \times J$. Since the cardinality of the left hand side equals $\sum_j \, |A_\alpha|$ and the cardinality of the right hand side equals $|X| \cdot |J|$, the conclusion follows immediately.■