Math 144
Fall 2019

UPDATED GENERAL INFORMATION — DECEMBER 4, 2019

Still more exercises

6. Let $\mathcal{E}$ be an equivalence relation on a set Y such that each equivalence class $A_\alpha \subset Y$ contains at least two elements. Show that there is a subset $S \subset Y$ such that for each equivalence class A_α the intersection $A_\alpha \cap S$ consists of exactly two elements. [*Hint:* This requires the Axiom of Choice.]

7. Let A and B be sets. Prove the following transfinite generalization of a familiar formula for finite sets: $|A \cup B| + |A \cap B| = |A| + |B|$.

Solutions to exercises in `aabNewUpdate12.144.d19.pdf`

0. The inclusion mapping $A \cap B \to A$ is 1–1, so by definition we have $|A \cap B| \leq |A|$.∎

1. (*a*) Suppose that $f(x) = f(y)$. The clearly $g(f(x)) = g(f(y))$. Since $g \circ f$ is 1–1, it follows that $x = y$ and hence that f is also 1–1. Now suppose $z \in C$. Since $g \circ f$ is onto we have $z = g(f(w))$ for some $w \in A$, and if $v = f(w)$ then we have $w = g(v)$, which means that g is onto.∎

(*b*) The relation $\mathcal{Y}$ is reflexive, for both $a\ \mathcal{V}\ b$ and $a\ \mathcal{W}\ b$ are true because $\mathcal{V}$ and $\mathcal{W}$ are equivalence relations. Similarly, $\mathcal{Y}$ is symmetric, for $a\ \mathcal{Y}\ b$ and $b\ \mathcal{Y}\ a$ implies both $a\ \mathcal{V}\ b$ and $b\ \mathcal{V}\ a$, and also $a\ \mathcal{W}\ b$ and $b\ \mathcal{W}\ a$; these combine to yield $b\ \mathcal{Y}\ a$. Finally, $\mathcal{Y}$ is transitive. If $a\ \mathcal{Y}\ b$ and $b\ \mathcal{Y}\ c$, then we have the following:

$a\ \mathcal{V}\ b$ and $b\ \mathcal{V}\ c$.

$a\ \mathcal{W}\ b$ and $b\ \mathcal{W}\ c$.

Since $\mathcal{V}$ and $\mathcal{W}$ are equivalence relations, these imply that $a\ \mathcal{V}\ c$ and $a\ \mathcal{W}\ c$. By definition these imply $a\ \mathcal{Y}\ c$. We have now shown that the new relation has the required properties of an equivalence relation on S.∎

2. This is false. In a well-ordered set every nonempty subset has a least element. This does not hold for $\mathbb{N}_-$ itself, for if $x \in \mathbb{N}_-$ then we have $x - 1 < x \in \mathbb{N}_-$. Hence $\mathbb{N}_-$ has no least element, and therefore the set is not well-ordered.∎

3. Split the set A of open intervals into four pieces B, C, D, E, where B is the set of intervals for which both endpoints are finite, C is the set where only the right hand endpoint is finite, D is the set where only the left hand endpoint is finite, and E is the set where neither endpoint is finite.

The intervals in B are determined by the ordered pair of endpoints (a, b) where $a < b$, so the number of such intervals is $\leq |\mathbb{R} \times \mathbb{R}| = 2^{\aleph_0} \times 2^{\aleph_0} = 2^{\aleph_0}$. The intervals in C and D are determined by their unique finite endpoints, so we have $|C| = |D| = 2^{\aleph_0}$. Finally, there is only one interval in E; namely, $\mathbb{R} = (-\infty, \infty)$. If we combine these we see that

$$2^{\aleph_0} \ = \ |C| \ \leq \ |A| \ + \ |B| + |C| + |D| + |E| \ =$$

$$2^{\aleph_0} + 2^{\aleph_0} + 2^{\aleph_0} + 1$$

and the result follows from the Schröder-Bernstein Theorem because this sum is merely $2^{\aleph_0}$.■

4. Let $\mathbf{P}(n)$ be the summation formula for each $n \geq 2$. Then one can check directly that if $n = 2$ both sides of the equation simplify to $\frac{1}{2}$. Suppose now that $\mathbf{P}(n)$ is known to be true for $n \geq 2$. Then we have

$$\sum_{k=2}^{n+1} \frac{1}{k^2 - k} \quad = \quad \sum_{k=2}^{n} \frac{1}{k^2 - k} \; + \; \frac{1}{(n+1)^2 - n + 1} \quad = \quad (\text{by } \mathbf{P}(n)\,)$$

$$1 \; - \; \frac{1}{n} \; + \; \frac{1}{n(n+1)} \quad = \quad 1 \; - \; \frac{n+1}{n(n+1)} \; + \; \frac{1}{n(n+1)} \quad = \quad 1 \; - \; \frac{1}{n+1}$$

which is exactly the equation in the formula $\mathbf{P}(n+1)$. Therefore $\mathbf{P}(n)$ implies $\mathbf{P}(n+1)$ for all $n \geq 1$, and by the Weak Principle of Finite Induction the statements $\mathbf{P}(n)$ are true for all $n \geq 2$.■

5. By definition $C + D = (C - D) \cup (D - C)$. Since taking inverse images commutes with unions, intersections and complements it follows that

$$f^{-1}[C + D] \quad = \quad f^{-1}[(C - D) \cup (D - C)] \quad =$$

$$f^{-1}[C - D] \cup f^{-1}[D - C] \quad = \quad \left(f^{-1}[C] - f^{-1}[D]\right) \; \cup \; \left(f^{-1}[D] - f^{-1}[D]\right)$$

where the right hand side equals $f^{-1}[C] + f^{-1}[D]$ by the definition of the latter.■

6. If $\mathbf{P}(Y)$ denotes the set of all subsets of Y, then the Axiom of Choice implies that there is a (choice) function $k : \mathbf{P}(Y) - \{\emptyset\} \longrightarrow Y$ such that $k(B) \in B$ for each nonempty subset $B \subset A$. Define a function c_1 from the set of equivalence classes $\{A_\alpha\}$ to Y such that $c_1(A_\alpha) = k(A_\alpha) \in A_\alpha$ for each A_α.

Since each set A_α has at least two elements, it follows that each set $A_\alpha - \{c_1(A_\alpha)\}$ is nonempty. Now define c_1 from the set of equivalence classes $\{A_\alpha\}$ by $c_2(A_\alpha) = k(A_\alpha - \{c_1(A_\alpha)\}) \in A_\alpha - \{c_1(A_\alpha)\} \subset A_\alpha$ for each A_α.

By construction $c_1(A_\alpha) \neq c_2(A_\alpha)$, and both of these elements lie in A_α. Furthermore, if $\beta \neq \alpha$ then $A_\alpha \cap A_\beta = \emptyset$ implies that neither $c_1(A_\beta)$ nor $c_2(A_\beta)$ belongs to A_α. Therefore if S is the set of all points $c_j(A_\gamma)$, where $j = 1$ or 2 and A_γ runs through all equivalence classes, then for each equivalence class A_α the intersection $S \cap A_\alpha$ consists of exactly two points.■

7. By definition, $|A| + |B|$ is the cardinal number of $(A \times \{1\}) \cup B \times \{2\}$, and hence it is the sum of the cardinal numbers

$$|(A - B) \times \{1\}| \; + \; |(A \cap B) \times \{1\}| \; + \; |(B - A) \times \{2\}| \; + \; |(A \cap B) \times \{2\}| \; .$$

The sum of the first three terms is $|A \cup B|$ and the fourth term is equal to $|A \cap B|$.■

This transfinite formula is far less useful than its finite counterpart because there is no consistent way to define subtraction in the transfinite case.