Math 144
Fall 2019

UPDATED GENERAL INFORMATION — DECEMBER 5, 2019

Solutions to quizzes

Quiz 2 Problem. Prove by induction that $3^{n+7} < (n+7)!$ for all nonnegative integers n.

You may use the identities $3^6 = 729$ and $6! = 720$ without verifying them. Also, recall that if $n > 0$ then $n!$ is the product of all the positive integers from 1 to n (so the recursive definition is $(n+1)! = n!(n+1)$).

Solution. Let $\mathbf{P}(n)$ denote the statement $3^{n+7} < (n+7)!$. We first verify that $\mathbf{P}(0)$ is true; it is only necessary to note that $3^7 = 3^6 \cdot 3 = 729 \cdot 3 = 2187$ and $7! = 6! \cdot 7 = 720 \cdot 7 = 5040$. Suppose now that $\mathbf{P}(n)$ is true for some $n \geq 0$. Then we can derive $\mathbf{P}(n+1)$ from the following chain of equations and inequalities:

$$3^{n+8} \; = \; 3 \cdot 3^{n+7} \; < \; (\text{by } \mathbf{P}(n)) \; 3 \cdot (n+7)! \; < \; (n+8) \cdot (n+7)! \; = \; (n+8)!$$

Therefore the Weak Principle of Finite Induction implies that each statement $\mathbf{P}(n)$ is true.■

Quiz 3 Problem. Letα, β, γ be nonzero cardinal numbers such that α is infinite and $\beta < \gamma$. Show that $\alpha \cdot \beta \leq \alpha \cdot \gamma$ and give an example to show that $\alpha \cdot \beta$ is not necessarily (strictly) less than $\alpha \cdot \gamma$.

Solution. Let A, B, C be sets such that $|A| = \alpha$, $|B| = \beta$ and $|C| = \gamma$. Then $\beta < \gamma$ implies that there is a 1–1 mapping $g : B \to C$ (which cannot be onto since the inequality is strict). If we define $G : A \times B \to A \times C$ by $G(a, b) = (a, g(b))$, then one can check directly that G is 1–1; specifically, $G(a, b) = G(a', b')$ implies $a = a'$ and $g(b) = g(b')$, and since g is 1–1 it follows that $b = b'$. This means that

$$\alpha \cdot \beta \quad = \quad |A \times B| \quad \leq \quad |A \times C| \quad = \quad \alpha \cdot \gamma \; .$$

To show that the inequality need not be strict, let $\alpha = \aleph_0 = \gamma$ and $\beta = 1$. Then clearly $\beta < \gamma$, but $\alpha \cdot \beta = \aleph_0 = \aleph_0 \cdot 1 = \aleph_0 \cdot \aleph_0 = \beta \cdot \gamma$, and since $\aleph_0 \cdot \aleph_0 = \aleph_0$ it follows that in this example we have $\alpha \cdot \beta = \alpha \cdot \gamma$.■