Math 144
Fall 2017

UPDATED GENERAL INFORMATION — DECEMBER 4, 2017

The second in-class examination

The second in-class examination will take place in the lecture sections on Friday, December 8. It will consist of four problems, just like the first examination. A few practice problems are given below; they cover more material than can be included in the examination. **Reminder.** The material to be covered on the second examination is given in the document `exam2review1.pdf`, which has been updated to list subsections that will NOT be covered.

The course ends with the completion of the second examination. There will be no examinations or class meetings during the week of December 11. The results of the second examination will be posted after grading and final grade assignments are complete.

1. Show that the only binary relation on a set which is both an equivalence relation and a partial ordering is the equality relation (a and b are related if and only if $a = b$).

2. Consider the divisibility relation on the set $\mathbb{N}_+$ of positive integers, which is defined by $a|b$ if and only if $b = ca$ for some $c \in \mathbb{N}_+$. Verify that this defines a partial ordering which is not a linear ordering.

3. Determine the number of distinct equivalence relations on the set $\{1, 2, 3, 4\}$ and also the number of **all** binary relations on that set.

4. Consider the binary relation on the real numbers $\mathbb{R}$ defined by the equation $x^2 \leq y^2$. Is this a partial ordering? Either prove it is or show that it is not.

5. (a) Let $f : A \to B$ and $g : B \to A$ be functions such that $f \circ g$ is the identity on B. Prove that g is 1–1 and f is onto, and construct an example such that f is not 1–1 and g is not onto.

(b) Let $f : A \to B$ and $g : B \to A$ be functions such that $f \circ g \circ f = f$ (note that f satisfies this condition if it satisfies the condition in part (a) of the exercise). Give examples to show that f need not be 1–1 or onto.

6. Prove by Mathematical Induction that if $n \geq 6$ then $n^2 > 4n + 2$.

7. Prove by Mathematical Induction that if $n \geq 1$ then $3^n > n^2 + 1$.

8. Define a sequence $\{a_n\}$ recursively by the formula $a_0 = 1$ and $a_n = 3 - a_{n-1}$ if $n \geq 1$. Prove by Mathematical Induction that $a_n = 2$ if n is odd and 1 if n is even.

9. Define a sequence $\{a_n\}$ recursively by the formula $a_1 = a_2 = 1$ and $a_n = 3a_{[n/3]}$ for all $n \geq 3$, where $[x]$ denotes the greatest integer $\leq x$. Prove that $a_n \leq n$ for all $n \geq 1$.

10. Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ denote the set of all functions from $\mathbb{R}$ to itself; note that such functions need not be continuous anywhere. If $\mathbf{c}$ denotes the cardinality of the real numbers, show that $|\mathcal{F}(\mathbb{R}, \mathbb{R})| = 2^{\mathbf{c}}$. [*Hints:* The cardinal number on the right is equal to $|\mathcal{F}(\mathbb{R}, \{0, 1\})|$ since $\{0, 1\} \subset \mathbb{R}$. What inequality does that imply? For the other inequality, recall that by definition a function from $\mathbb{R}$ to itself is completely determined by its graph, which is a subset of $\mathbb{R} \times \mathbb{R}$].

11. Let α be a transfinite cardinal number such that $\alpha \cdot \alpha = \alpha$, and let $\mathcal{E}$ denote the set of finite subsets of a set with cardinality α. Prove that the cardinality of $\mathcal{E}$ is also α.

12. Prove that the hypothesis in the first sentence of Exercise 11 is valid if $\alpha = 2^{\mathbf{c}}$. [*Hint:* Recall that $\mathbf{c} + \mathbf{c} = \mathbf{c}$.]