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Mathematics 144, Winter 2017, Examination 1

Answer Key

1. [20 *points*] Let X be a set, and let A and B be subsets of X. Prove that

$$A \;\cap\; ((X - A) \cup B) \;\;=\;\; A \;\cap\; B\;.$$

SOLUTION

By the Distributive Law for an intersection with a union, we have

$$A \;\cap\; ((X - A) \cup B) \;\;=\;\; (A \cap (X - A)) \;\cup\; (A \cap B)\;.$$

Now $A \cap (X - A) = \emptyset$, so the right hand side reduces to $\emptyset \cup (A \cap B) = A \cap B$.∎

2. [30 *points*] (a) Let N be the binary relation on the real numbers $\mathbb{R}$ defined by xNy if and only if $|x-y| < 1$. Determine if N is (i) reflexive, (ii) symmetric, (iii) transitive, giving reasons to support your conclusions in all three instances.

(b) Let X, A, B be sets. Prove that

$$(A \times X) \ \cup \ (B \times X) \quad = \quad (A \cup B) \times X \ .$$

SOLUTION

(a) N is reflexive because $|x-x| = 0 < 1$. Since $|y-x| = |x-y|$ it follows that if xNy, then $|x-y| < 1$ implies $|y-x| = |x-y| < 1$, so that yNx. However, N is not transitive. Geometrically this may seem clear, but to verify this we need to give explicit examples. If we take $x = 0, y = \frac{3}{4}, z = \frac{3}{2}$, then xNy and yNz because $|x-y| = |y-z| = \frac{3}{4} < 1$, but xNz is not true because $|x-z| = \frac{3}{2} > 1$.■

(b) Suppose that $(u,v) \in (A \times X) \cup (B \times X)$. Then in all cases we have $v \in X$, and we also have $u \in A$ or $u \in B$ depending upon whether the ordered pair lies in $A \times X$ or in $B \times X$. In either case $(u,v) \in (A \cup B) \times X$, and hence the left hand side of the displayed statement is contained in the right hand side, completing half of the argument.

Conversely, if $(u,v) \in (A \cup B) \times X$ then $u \in A \cup B$, and $v \in X$. There are two cases now, depending upon whether $u \in A$ or $u \in B$. In the first case $(u,v) \in A \times X$, and in the second case $(u,v) \in B \times X$; hence in either case $(u,v) \in (A \times X) \cup (B \times X)$, so that the right hand side of the displayed statement is contained in the left hand side. Combining this with the conclusion of the first paragraph, we see that the two sets must be equal.■

3. [*25 points*] (a) Let A be a set, and let R be an equivalence relation on A. Suppose we are given $a, b, c, d \in A$ such that aRb, cRb, and cRd are all true. Prove that dRa is also true.

(b) Give examples of two subsets A, B of the integers $\mathbb{Z}$ such that $A \times B \neq B \times A$.

SOLUTION

(a) If aRb and cRb are true, then aRb and bRc are true because R is symmetric, and therefore aRc is true by transitivity.

Since we now know aRc is true and cRd is assumed to be true, it follows that aRd is true by transitivity. Since R is symmetric, the last relation implies that dRa is also true.■

(b) Let A be the even integers and let B be the odd integers. Then the two products are unequal and in fact $(A \times B) \cap (B \times A) = \emptyset$.■

4. [25 *points*] Given a set S, let $\mathbf{P}(S)$ denote the set of all subsets of X. If A is a subset of the set X, then we know from the exercises that

$$\mathbf{P}(X - A) \; - \; \{\emptyset\} \quad \subset \quad \mathbf{P}(X) \; - \; \mathbf{P}(A) \ .$$

Find an example $\emptyset \neq A \subset X$ for which this containment is proper.

SOLUTION

We start by restating the goal, which is to find a set X with a nonempty subset A such that $(\star)$ *the set X has a subset B which is not contained in A or in $X - A$*. In other words, B should be a subset which contains points of both A and $X - A$. — In fact, if A is a nonempty proper subset of X, then $X \in \mathbf{P}(X) - \mathbf{P}(A)$ but X contains points of both A and $X - A$.$\blacksquare$

It would also suffice to give a specific counterexample. In particular, let $X = \{1, 2, 3\}$ and $A = \{1\}$. Then every subset $B \subset X$ with $1, 2 \in B$ lies in $\mathbf{P}(X) - \mathbf{P}(A)$ but not in $\mathbf{P}(X - A) - \emptyset$, for B contains points in both A and $X - A$.$\blacksquare$