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Mathematics 144, Fall 2017, Examination 2

Answer Key

(This incorporates the changes given in exam2af17.pdf)

1. [20 *points*] Let $\mathbb{R}[t]$ denote the set of polynomials in one ideterminate (namely, t). If we define a binary relation $\leq$ on $\mathbb{R}[t]$ by the condition $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, show that this defines a partial ordering of $\mathbb{R}[t]$ which is not a linear ordering.

SOLUTION

The relation is reflexive, for $f \leq f$ because $f(x) \leq f(x)$ for all x.

The relation is antisymmetric, for $f \leq g$ and $g \leq f$ imply $f(x) \leq g(x)$ and $g(x) \leq f(x)$ for all x. Therefore $f(x) = g(x)$ for all x because $\mathbb{R}$ is partially ordered.

The relation is transitive, for $f \leq g$ and $g \leq h$ imply $f(x) \leq g(x)$ and $g(x) \leq h(x)$, so that $f(x) \leq h(x)$ for all x and hence $f \leq h$. Combining the first three paragraphs, we see that the relation on polynomials is a partial ordering.

To see it is not a linear ordering, we need to find f and g so that neither $f \leq g$ nor $g \leq f$. We can do this if we find f and g such that $f(a) < g(a)$ but $g(b) < f(b)$ for some $a \neq b$ in $\mathbb{R}$. One simple example is $f(x) = x$ and $g(x) = 1 - x$ for $a = 0$ and $b = 1$.■

2. [30 *points*] (*a*) Suppose that $f : A \to B$ and $g : B \to A$ are functions such that the composite $g \circ f$ is the identity function on A. Prove that f is 1–1 and g is onto, and give examples f, g such that f is not onto and g is not 1–1. [*Hint:* For the second part, let $A \neq \emptyset$ be as small as possible.]

(*b*) Suppose that A is a finite set with $n > 0$ elements. Prove that there are 2^{n^2} distinct binary relations on A (*i.e.*, relating A to itself).

SOLUTION

(*a*) f is 1–1, for $f(a) = f(a')$ implies $a = g(f(a)) = g(f(a')) = a'$, and g is onto because $a \in A$ implies $a = g(b)$ where $b = f(a)$. The easiest negative examples are $A = \{0\}$ and $B = \{0, 1\}$ with $f(0) = 0$ (hence 1 is not in the image of f, so f is not onto) and $g(b) = 0$ for $b = 0, 1$ (hence g is not 1–1).■

(*b*) A binary relation is given by a subset of $A \times A$. If A has n elements, then $A \times A$ has n^2 elements and hence there are 2^{n^2} binary relations corresponding to the subsets in $A \times A$.■

3. [*30 points*] (*a*) Let $n!$ denote the product of the first n positive integers. Prove that $2^n < n!$ for all $n \geq 4$. [*Note:* The inequality fails for $n \leq 3$ but there is no need to prove this.]

(*b*) Suppose that $\{a_n\}$ is a sequence defined recursively by $a_1 = 1$ and $a_k = 2a_{[k/2]}$ for all $k \geq 2$, where $[x]$ denotes the greatest integer $\leq x$. Prove that $a_n \leq n$ for all integers $n \geq 1$.

SOLUTION

(*a*) If $n = 4$ this is true because $2^4 = 16 < 24 = 4!$. Suppose the result is valid for $n = m \geq 4$, and consider the stated inequality when $n = m + 1$. We then have $2^{m+1} = 2^m \cdot 2 < m! \cdot (m+1) = (m+1)!$, so the inequality for $n = m + 1$ is also valid, completing a proof by the Weak Principle of Finite Induction.

(*b*) It is straightforward to verify the inequality for $n = 1$ and $n = 2$. Suppose that the inequality is valid for all $n < m$, where $m \geq 3$, so that $m > [m/2] \geq 1$. We then have $a_{[m/2]} \leq [m/2] \leq m/2$, so that $2 \cdot a_{[m/2]} \leq m$, which is the inequality for $n = m$. Therefore the validity of the conjecture for $1 \leq n < m$ implies its validity for $n = m$, proving the result by the Strong Principle of Finite Induction.■

4. [20 *points*] Let $\mathcal{F}_2$ denote the family of all subsets in $\mathbb{R}^2$ which contain exactly TWO elements. Show that $\mathcal{F}_2$ and $\mathbb{R}$ have the same cardinality. [*Hint:* If $n \geq 2$ is an integer, how are the cardinalities of $\mathbb{R}^n$ and $\mathbb{R}$ related?]

SOLUTION

Given a subset $\{a, b\}$ in $\mathcal{F}_2$, write $a = (a_1, a_2)$ and $b = (b_1, b_2)$ and choose a linear ordering of $\{a, b\}$; reversing the roles of a and b if necessary, we might as well assume that a comes first. Using this ordering, define a 1–1 mapping $\mathcal{F}_2 \to \mathbb{R}^4$ sending $\{a, b\}$ to (a_1, a_2, b_1, b_2). We then have $|\mathcal{F}_2| \leq |\mathbb{R}^4| = |\mathbb{R}|$.

To prove the reverse inequality, given $t \in \mathbb{R}$ send it to the two point set $\{(t, 0), (0, 1)\} \subset \mathbb{R}^2$. This construction defines a 1–1 mapping from $\mathbb{R}$ to $\mathcal{F}_2$, so that $|\mathbb{R}| \leq |\mathcal{F}_2|$. We can now apply the Schröder-Bernstein Theorem to conclude that $\mathbb{R}$ and $\mathcal{F}_2$ have the same cardinality.$\blacksquare$