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Mathematics 144, Winter 2019, Examination 2

Answer Key

1. [30 *points*] (a) Let S denote the set of real valued sequences $\{x_n\}$ where $n \geq 0$ and $x_n \in \mathbb{R}$, and define a binary relation $\{x_n\}\mathcal{A}\{y_n\}$ if and only if there are only finitely many values of n such that $x_n \neq y_n$. Show that $\mathcal{A}$ defines an equivalence relation on S.

(b) Let X be a set, and let $f : X \to X$ be a map such that $f \circ f$ is 1–1 and onto. Prove that f is 1–1 and onto.

SOLUTION

(a) The relation is reflexive, for if $\{x_n\}$ is a sequence then $\{x_n\}\mathcal{A}\{x_n\}$ because $x_n = x_n$ for all n.

The relation is symmetric, for if $\{x_n\}$ and $\{y_n\}$ are sequences such that $x_n \neq y_n$ for only finitely many values of n, then $y_n \neq x_n$ for only the same finitely many values or n.

The relation is transitive. If $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences such that $x_n \neq y_n$ for only the values of n in the finite set F, and $y_n \neq z_n$ for only the values of n in the finite set F', then $x_n \neq z_n$ for only values of n in the finite set $F \cup F'$. There might be values of n in the latter set such that $x_n = z_n$, but if $x_n \neq z_n$ then $n \in F \cup F'$.■

(b) The function f is 1–1 because $f(u) = f(v)$ implies $f \circ f(u) = f \circ f(v)$, and since $f \circ f$ is 1–1 this implies $u = v$. The function f is also onto. Since $f \circ f$ is onto, for each $x \in X$ we have $x = f \circ f(y)$ for some $y \in X$. If $z = f(y)$, it follows that $x = f(z)$, and since x is arbitrary it follows that f is onto.■

2. [20 *points*] Prove the following formula; this can be done by mathematical induction:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} \;=\; 1 \;-\; \frac{1}{n+1}$$

SOLUTION

Let $\mathbf{P}(n)$ be the summation formula for each $n \geq 1$. Then one can check directly that if $n = 1$ both sides of the equation simplify to $\frac{1}{2}$. Suppose now that $\mathbf{P}(n)$ is known to be true for $n \geq 1$. Then we have

$$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} \;=\; \sum_{k=1}^{n} \frac{1}{k(k+1)} \;+\; \frac{1}{(n+1)(n+2)} \;=\; (\text{by } \mathbf{P}(n)\,)$$

$$1 \;-\; \frac{1}{n+1} \;+\; \frac{1}{(n+1)(n+2)} \;=\; 1 \;-\; \frac{n+2}{(n+1)(n+2)} \;+\; \frac{1}{(n+1)(n+2)} \;=\; 1 \;-\; \frac{1}{n+2}$$

which is exactly the equation in the formula $\mathbf{P}(n+1)$. Therefore $\mathbf{P}(n)$ implies $\mathbf{P}(n+1)$ for all $n \geq 1$, and by the Weak Principle of Finite Induction the statements $\mathbf{P}(n)$ are true for all $n \geq 1$.■

3. [*25 points*] (*a*) Suppose that A and B are sets. Show that their cardinalities satisfy $|A \cup B| \leq |A| + |B|$.

(*b*) Give two examples of infinite sets A, B such that $|A| \neq |B|$ but $|A|, |B| > \aleph_0$.

SOLUTION

(*a*) The right hand side equals the cardinality of the disjoint union $A \times \{1\} \cup B \times \{2\}$, so it suffices to define a 1–1 map h from $A \cup B$ to the latter. One way of doing this is to set $h(x) = (x, 1)$ if $x \in A$ and $h(x) = (x, 2)$ if $x \in B - A$.■

Alternate solution. Define a map $k : A \times \{1\} \cup B \times \{2\} \longrightarrow A \cup B$ by $k(x, t) = x$, where $x \in A \cup B$ and $t \in \{1, 2\}$. This map is onto; if $x \in A$ then $x = k(x, 1)$, while if $x \in B$ then $x = k(x, 2)$. By a proposition on cardinal numbers (which is related to the Axiom of Choice for uncountable sets) it follows that $|A| + |B| \geq |A \cap B|$.■

(*b*) If $A = \mathbb{R}$, then $|A| = 2^{\aleph_0} > \aleph_0$, and if B is the set of all subsets of $\mathbb{R}$ then $|B| = 2^{|A|} > |A| > \aleph_0$.■

4. [25 *points*] (a) Explain why the set $\mathbb{Q}_+$ of positive rational numbers is not well-ordered.

(b) State the Axiom of Choice.

SOLUTION

(a) In a well-ordered set every nonempty subset has a least element. However $\mathbb{Q}_+$ itself has no least element, for if $a > 0$ is a positive rational number then so is $\frac{1}{2}\,a$, and we have $0 < \frac{1}{2}\,a < a$. Therefore $\mathbb{Q}_+$ is not well-ordered.■

(b) Let X be a set, and let $\mathbf{P}_0(X)$ denote the set of nonempty subsets of X. Then there is a function $c : \mathbf{P}_0(X) \longrightarrow X$ such that $c(A) \in A$ for each nonempty subset $A \subset X$.■