Practice for Exam 2

Problems from `aabUpdate09.144.f17.pdf`

1. Let $\mathcal{E}$ be a relation which is both a partial ordering and an equivalence relation. Since we have an equivalence relation, $a\ \mathcal{E}\ b$ implies $b\ \mathcal{E}\ a$. However, since $\mathcal{E}$ is also a partial ordering these conditions imply that $a = b$. This is almost but not quite enough, for we need to check that $a\ \mathcal{E}\ a$ for all a in the set. But the latter property holds because both partial orderings and equivalence relations are reflexive.■

2. The relation is reflexive because $a = 1 \cdot a$. If $a|b$ and $b|a$ then there are positive integers x and y so that $b = xa$ and $a = yb$; combining these, we have $a = (yx)a = 1 \cdot a$. The latter implies that $xy = 1$, which in turn implies that $x = y = 1$ since both x and y are positive integers. Hence $a = b$, showing that the relation is symmetric. To prove transitivity, note that $a|b$ and $b|c$ imply $b = xa$ and $c = yb$, so that $c = (xy)b$; but the latter implies $a|c$ and hence shows that the relation is transitive.

To see that the ordering is not linear, it suffices to note that 2 does not (evenly) divide 3 and 3 does not (evenly) divide 2.■

3. The number of equivalence relations of $\{1, 2, 3, 4\}$ is equal to the number of partitions of that set. We can classify the partitions into types, listed in order of the sizes of the largest subset:

(1) 4

(2) 3+1

(3) 2+2

(4) 2+1+1

(5) 1+1+1+1

There is only one partition of the first type, there are four of the second type (depending upon which element is in the one element partition), there are three of the third type (each subset of $\{1, 2, 3, 4\}$ with two elements determines a partition, complementary subsets determine the same partition, and there are six subsets with two elements), there are six partitions of the fourth type (one for each subset with two elements; the choice of this set dictates the remaining possibilities), and there is only one partition of the fifth type. Therefore there are a total of $1+4+3+6+1 = 15$ partitions for the set $\{1, 2, 3, 4\}$.■

The number of binary relations on a set A is the number of elements in $A \times A$ (if A is finite), and hence is $\left(2^{|A|^2}\right)$. If $|A| = 4$, then this number is $2^{16} = 65,536$.■

4. The binary relation is not symmetric, for $|x|^2 \leq |y|^2$ and $|y|^2 \leq |x|^2$ implies $|x|^2 = |y|^2$, which is true if and only if $y = \pm x$. Hence we have $-1\ \mathcal{R}\ 1$ and $1\ \mathcal{R}\ -1$ even though $-1 \neq 1$.■

5. (a) Let B consist of a single point p, and let A consist of more than one point. Take $g : B \to A$ so that $g(a) = b \in B$ is the unique point and $f(b) = a$, where a is an arbitrary point. Then f is not onto and g is not 1–1; both of these follow because $|B| > 1$.■

(b) Let $f : \{1, 2\} \to \{1, 2\}$ send everything to 1, and let g also be the map sending everything to 1. Then f is neither 1–1 nor onto, and the same is true for g, but we have $f \circ g \circ f(n) = 1 = f(n)$ for $n = 1, 2$.■

6. If $n = 6$ then the result is true because $n^2 = 36 > 26 = 4n + 2$. Suppose the inequality is true for $n = m \geq 6$. To prove the result when $n = m + 1$, we shall show that $(m + 1)^2 - m^2 > (4(m + 1) + 2) - (4m + 2)$ if $m \geq 6$; if the latter is true then we can add it to the inductive hypothesis $m^2 > 4m + 2$, and this will prove the validity of the inequality for $n = m + 1$. Now $(4(m+1)+2)-(4m+2) = 4$, and we have $(m+1)^2 - m^2 \;=\; 2m+1$, which is greater than 12 since $m \geq 6$. Therefore we have established the inequality when $n = m + 1$, completing the proof by induction.■

7. The inequality is valid when $n = 1$ because $3^1 = 3 > 2 = 1^2 + 1$. Now suppose that the inequality is valid when $n = m$, where $m \geq 1$. As in the preceding exercise, the verification of the inequality when $n = m + 1$ reduces to showing $3^{m+1} - 3^m > ((m + 1)^2 + 1) - (m^2 + 1) = 2m + 1$. Now the left hand side equals $2 \cdot 3^m$. By the validity of the inequality when $n = m$, this is greater than $2 \cdot (m^2 + 1) = 2m^2 + 2 \geq 2m + 1$ because $m \geq 1$. Therefore the validity of the inequality for $n = m$ implies its validity for $n = m + 1$, completing the proof by induction.■

8. Direct computation implies that $a_2 = 2$. Suppose now that the formula is valid for all $n = m - 1$, where $m \geq 3$. We now need to verify the inequality for $n = m \geq 3$; in this case $m - 1 \geq 1$ so the induction hypothesis implies that $a_{m-1} = 1$ if $m - 1$ is even and 2 if $m - 1$ is odd. It follows that $a_m = 3 - a_{m-1}$ is equal to 2 if $m - 1$ is even and 1 if $m - 1$ is odd. Since a_m is even if and only if a_{m-1} is odd, and similarly if odd and even are switched, it follows that $a_m = 1$ if m is even and 2 if m is odd. This completes the proof of the inductive step.■

9. The result is true if $n = 1, 2$ by assumption. Assume it is true for $n < m$. Then $a_m = 3a_{[m/3]}$ by definition, and by the induction hypothesis we know that the right hand side is $\leq 3 \cdot [m/3] \leq 3 \cdot (m/3) \leq m$. This verifies the inductive step, and it follows that the result is true by the Strong Principle of Finite Induction.■

10. Follow the hints. Since $\{0, 1, \} \subset \mathbb{R}$ it follows that there is a 1–1 correspondence between $\mathcal{F}(\mathbb{R}, \{0, 1\})$ and a subset of $\mathcal{F}(\mathbb{R}, \mathbb{R})$, and therefore we have $2^{\mathbf{c}} \leq |\mathcal{F}(\mathbb{R}, \mathbb{R})|$. If we can also prove the reverse inequality, then the cardinalities will be the same by the Schröder-Bernstein Theorem.

A function from $\mathbb{R}$ to itself is completely determined by its graph, which is a subset of $\mathbb{R} \times \mathbb{R}$. Therefore if $\mathbf{c} = |\mathbb{R}|$ we have $|\mathcal{F}(\mathbb{R}, \mathbb{R})| \leq 2^{\mathbf{c} \times \mathbf{c}}$. But we have seen that $\mathbf{c} \times \mathbf{c} = \mathbf{c}$, so the right hand side is just $2^{\mathbf{c}}$. Hence $2^{\mathbf{c}} = |\mathcal{F}(\mathbb{R}, \mathbb{R})|$ by the Schröder-Bernstein Theorem.■

11. If $|X| = \alpha$ and $A \subset X$ has n elements, pick an ordering for the elements of A having the form $a_1, \;\cdots\;, a_n$, and send A to $(a_1, \;\cdots\;, a_n; n) \in X^n \times \{n\}$. This gives a 1–1 map from the set $\mathbf{P}_n(X)$ of subsets with n elements to the set $A^n \times \{n\}$; since $\alpha \cdot \alpha = \alpha$, BY induction we also have $\alpha^n = \alpha$ for all positive integers n, and thus we have a 1–1 mapping from $\mathbf{P}_n(X)$ to $X \times \{n\}$. These maps piece together to yield a 1–1 mapping from $\mathcal{E}$ to $\mathbb{N} \times X$. This implies that $|\mathcal{E}| \leq \aleph_0 \cdot \alpha \leq \alpha \cdot \alpha = \alpha$. The reverse inequality $\alpha \leq |\mathcal{E}|$ follows because the map sending $x \in X$ to $\{x\}$ is a 1–1 map from X to $\mathcal{E}$.■

12. The laws of exponents for transfinite cardinal numbers implies that $2^{\alpha+\beta} = 2^\alpha \cdot 2^\beta$. If $\alpha = \beta = \mathbf{c}$, then this specializes to $2^{\mathbf{c}} = 2^{\mathbf{c}+\mathbf{c}} = 2^{\mathbf{c}} \cdot 2^{\mathbf{c}}$.■