ADDITIONAL EXERCISES FOR

MATHEMATICS 144 — PART 2

Fall 2017

IV. Relations and functions

IV.1. Binary relations

101. Find the mistake in the following attempt to prove that the reflexive property of an equivalence relatin is redundant:

> Let $\mathcal{R}$ be a binary relation on S which is symmetric and transitive, and let $x \in S$. Then by the symmetric property $x\,\mathcal{R}\,y$ implies $y\,\mathcal{R}\,x$. The latter two combine with the transitivity property to imply that $x\,\mathcal{R}\,x$.

102. Let $\mathcal{R}$ be the binary relation on $\{a, b, c, d, e\}$ given by $\{(a, c), (b, d), (c, a), (d, e)\}$. Determine the equivalence classes for the equivalence relation $\mathcal{R}^{\#}$ generated by $\mathcal{R}$.

103. The game of chess is played on an 8×8 board with squares alternately colored black and white (or some other pair of contrasting colors). A chess player is likely to notice very quickly that a bishop can move to any square of the same color it currently occupies but cannot more to a square of the opposite color. The goal of the exercise is to give a mathematical proof of this assertion.

We begin by describing the situation in set-theoretic terms: Model the chessboard mathematically by the set

$$B \quad = \quad \{1, 2, 3, 4, 5, 6, 7, 8\} \times \{1, 2, 3, 4, 5, 6, 7, 8\}$$

so that the squares correspond to ordered pairs of points (i, j) and the color of a square depends upon whether $i+j$ is even or odd (each square which is one over to the left or right, or one above or one below, of a given square has the opposite color from the given square). Define a binary relation $\mathcal{R}$ on B such that $(i, j)\ \mathcal{R}\ (p, q)$ if $p = i + \alpha$ and $q = j + \beta$ where $\alpha, \beta \in \{-1, 1\}$ and $(p, q) \in B$ (these correspond to a bishop moving one square in any permissible direction on an empty board), and let $\mathcal{E}$ be the equivalence relation generated by $\mathcal{R}$.

This leads to a formal restatement of the exercise: Prove that $\mathcal{E}$ has exactly two equivalence classes, so that the equivalence class of a point is determined by whether $i + j$ is even or odd.

104. Suppose that $\mathcal{R}_1$ is an equivalence relation on X, let $X/\mathcal{R}_1$ denote the set of equivalence classes for $\mathcal{R}_1$, and let $\mathcal{R}_2$ be an equivalence relation on $X/\mathcal{R}_1$. Define a binary relation $\mathcal{S}$ on X such that $x\ \mathcal{S}\ y$ if and only if the equivalence classes $[x]$ and $[y]$ of $x, y \in X$ with respect to $\mathcal{R}_1$ satisfy $[x]\ \mathcal{R}_2\ [y]$. Prove that $\mathcal{S}$ also defines an equivalence relation on X.

IV.2. Partial and linear orderings

101. A partially ordered set $(X, \leq)$ is said to have a *greatest element* a if $a \geq x$ for all $x \in X$, and it is said to have a *least element* b if $b \leq x$ for all $x \in X$.

(a) Prove that if a and a' are greatest elements of X, then $a = a'$ (hence there is at most one greatest element), and similarly prove that if b and b' are least elements of X, then $b = b'$.

(b) Give examples of finite partially ordered sets $(X, \leq)$ such that (i) there is a greatest element and a least element, (ii) there is a greatest element but no least element, (iii) there is a least element but no greatest element, (iv) there is neither a greatest element nor a least element.

102. (a) Prove that every finite partially ordered set has at least one maximal element and at least one minimal element, and give an example of an infinite linearly ordered set which has neither type of element.

(b) Show that if X is a linearly ordered set then X has at most one maximal element and at most one minimal element.

103. (a) Give an example of a partially ordered set X and an $x \in X$ such that (i) the element x has more than one immediate successor, (ii) the element x has more than one immediate predecessor.

(b) If X is a linearly ordered set, prove that if an element x has an immediate successor s or an immediate predescessor p, then it has a unique immediate successor or immediate predescessor (respectively).

IV.3. Functions

101. A set J is called an *initial object* if for each set X there is a unique function $f : J \to X$, and a set T is called a *terminal object* if for each set X there is a unique function $g : X \to T$. Prove that the empty set is the only initial object and the terminal objects are precisely the one point sets of the form $\{p\}$ for some p.

102. Given two sets A and B, their *disjoint union* or *abstract sum* $A \amalg B$ is given by

$$A \amalg B \quad = \quad A \times \{1\} \;\cup\; B \times \{2\} \quad \subset \quad (A \;\cup\; B) \;\times\; \{1, 2\}$$

so that $A \amalg B$ is a union of two disjoint subsets, one of which is in 1–1 correspondence with A and the other of which is in 1–1 correspondence with B (see the comments below regarding the choice of symbols).

(i) If C is a third set, describe a 1–1 correspondence from $(A \amalg B) \times C$ to $(A \times C) \amalg (B \times C)$. [*Hint:* The left hand side is a subset of $(A \cup B) \times \{1, 2\} \times C$, and the right hand side is a subset of $(A \cup B) \times C \times \{1, 2\}$.]

(ii) If X is another set and $f : A \to X$, $g : B \to X$ are functions, prove that there is a unique function $h : A \amalg B \to X$ such that $h(a, 1) = f(a)$ for all $a \in A$ and $h(b, 2) = g(b)$ for all $b \in B$.

Remark on the notation. The disjoint symbol $\amalg$ is an upside down upper case Greek letter Pi ($= \Pi$). One of the reasons for this choice of symbols is that this construction can be viewed as a "dual" to the Cartesian product, which is denoted by Π, and another is that , $\amalg$ is similar but not identical to the usual symbol $\cup$ for the union of two sets.

IV.4. Composite and inverse functions

101. Let $f : \mathbb{R} \to \mathbb{R}$ denote the function $f(x) = \sin x$. Describe the following sets in terms of ordinary intervals; in other words, find the endpoints of the intervals and determine which endpoints are contained in the given intervals:

(a) $f[A]$, where $A = [0, \frac{1}{2}\pi]$.
(b) $f[A]$, where $A = [0, \infty)$.
(c) $f^{-1}[A]$, where $A = [0, \frac{1}{2}]$.
(d) $f^{-1}[A]$, where $A = [-1, 1]$.

102. Let $f : \mathbb{R} \to \mathbb{R}^2$ denote the function $f(x) = (x, 2x)$. Describe each of the following sets:
(a) $f[D]$, where $D = [0, 1]$.
(b) $f^{-1}[D]$, where $D = [0, 1] \times [0, 1]$.
(c) $f^{-1}[D]$, where $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

103. (a) Show that a map $f : X \to Y$ is onto if and only if $f\left[f^{-1}[C]\right] = C$ for all $C \subset Y$.
(b) Show that a map $f : X \to Y$ is 1–1 if and only if $f^{-1}\left[f[A]\right] = A$ for all $A \subset X$.

104. Let $f : X \to Y$ be a map, and let A and B be subsets of X. Prove that $f[A - B] = f[A] - f[B]$ if and only if $f[A - B] \cap f[B] = \emptyset$. Using this, prove that $f[A - B] = f[A] - f[B]$ if f is 1–1.

105. Let $f : X \to Y$ be a function, and suppose that $A \subset B \subset X$ and $C \subset D \subset Y$. Prove that $f[A] \subset f[B]$ and $f^{-1}[C] \subset f^{-1}[D]$.

106. Let A and B be subsets of a set X. Prove that there is a 1–1 onto function from A to the disjoint union $(A - B) \amalg (A \cap B)$. The disjoint union is defined in Exercise 102 for the preceding section.

IV.4. Constructions involving functions

101. Let A, B, C be sets. Prove that there are 1–1 correspondences $(B \times C)^A \leftrightarrow (B)^A \times (C)^A$ and $C^{A \amalg B} \leftrightarrow C^A \times C^B$. [*Hint:* Look at the coordinate projections onto the B and C factors in the first case and the restrictions to A and B in the second.]

IV.6. Order types

101. (a) Let p and q be distinct positive prime numbers, let $D(p, q)$ be the set of all positive integers which divide p^2q^2, and take the partial ordering $a|b$ given by the first number dividing the second. Prove that for evert other pair of distinct positive prime numbers p' and q' the partially ordered sets $D(p, q)$ and $D(p, q)$ have the same order type. (b) Let p and q be as above, and let $E(p, q)$ be the set of all positive integers which divide p^3q, with the partial ordering $a|b$ given by the first number dividing the second. Explain why $D(p, q)$ and $E(p, q)$ do not have the same order type.

102. For partially ordered sets with exactly three elements, there are precisely 5 order types (compare Problem 7.68 in Lipschutz; the solution for that exercise overlooks one possibility — namely, a partial ordering in which $a, b << c$ but there is no order comparison for a and b). Prove that there are precisely 16 order types for partially ordered sets with exactly four elements. [*Hints:* A few examples are illustrated in Problem 7.69 of Lipschutz. One systematic approach to this problem is to define a concept of **level** for an element of a partially ordered set: An element x is said to have level at least n if there is a sequence of elements $x = y_0, y_1, \cdots, y_n$ such that for each $1 \leq k \leq n$ the element y_k is an immediate predecessor of y_{k-1}, and x is said to have level equal to n if it has level at least n but does not have level at least $n + 1$. One can then split the family of partially ordered sets into categories depending on the largest L such that there is an element with level equal to n for various integers n.]

Note. For partially ordered sets with exactly five elements, there are precisely 55 order types. Clearly the number of order types will increase rapidly with the number of elements in a finite set.

103. Let X be a partially ordered set with ordering $\leq$. Prove that X has the same order type as some subset of $\mathcal{P}(X)$. Note that if $(S, \leq)$ is a partially ordered set and $T \subset S$, then the restriction of the partial ordering on S defines a partial ordering on T.